J. Phys. A: Math. Gen. **17** (1984) 1829-1842. Printed in the UK

# Tunnel transitions and vacuum polarisation in the potential well under the influence of an electric field[†]

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Received 30 March 1983

**Abstract** The exact solution of the problem of the vacuum polarisation and tunnel transitions of an electron out of the potential well under the influence of an electric field is obtained. The conclusion drawn earlier concerning the existence of the intense vacuum polarisation mechanism by the external field, for which the number of electron-positron pairs created in the field is proportional to the energy levels width of elementary electron-positron excitations, is confirmed. The estimates of the pair creation probability allow us to think that the atom ionisation in a steady electric field is accompanied by creation of electron-positron pairs which can be quite well registered experimentally at the relatively weak fields. It is shown that at the sudden switching on of an electric field, the value of the tunnel current of electron emission out of the well is affected appreciably by allowance for the positron band.

## 1. Introduction

In this paper the non-stationary problem of the vacuum polarisation and tunnel transitions out of the potential well under the influence of an electric field $\mathscr{E}$ switched on instantaneously at some moment of time is considered. The tunnel current occurring both in the one-electron state which corresponds to the bound electron state in the potential well before switching on the field $\mathscr{E}$ and in the vacuum state is calculated.

The electron tunnelling out of the potential well and the creation of electron-positron pairs are due to the decay of quasistationary states being formed in an electric field. When considering these phenomena we use the Drukarjev's method ( Drukarjev 1951 ), which offers the most rigorous and consistent way of describing the decay processes.

As is seen from a comparison of the results obtained and the non-relativistic tunneling theory results ( Oleinik and Arepjev 1983 ), when considering the electron tunnelling out of the well in the case of sudden switching on of an electric field it is important to take into consideration the positron band which markedly

[†] The main ideas of the present paper are briefly outlined in Oleinik and Belousov ( 1983 )

affects the value of tunnel current. For the one-level well the maximum value of tunnel current is considerably larger than that in the non-relativistic theory.

The exact solution of the vacuum polarisation problem given in the paper confirms the conclusion drawn earlier ( Oleinik 1981 ) concerning the existence of the pair creation mechanism for which the number of pairs created in an external field is proportional to the energy level width of elementary excitations of the electron-positron field. According to numerical estimates presented, the vacuum polarisation due to the appearance and decay of quasi-stationary states is considerably stronger than that caused by the steady and uniform electric field in the absence of the potential well ( Schwinger 1951 ).

In § 2 the general formulae for the electron current density in one-electron and vacuum states are given. In § 3 the correction to the energy and the width $\Gamma$ of the quasi-stationary electron level in the well are calculated. The formula for the electric current of electron-positron pairs created in an electric field is derived. The calculation of the electron tunnel current out of the well is given in § 4. The formulae for the wavefunctions of a relativistic electron in the well and in an electric field are presented in the appendix.

## 2. General formulae for the electric current density

Let us consider the electron-positron in the potential well

$$V_0(z) = -V_0\theta(z+L)\theta(-z), \qquad V_0 > 0 \tag{1}$$

where $V_0$ and $L$ are the depth and the width of the well. The electron-positron field operator of the system in the Schrödinger picture may be presented in the form ( for simplicity, we confine ourselves to investigating the one-dimensional model ):

$$\Psi_0(z,t) = \sum_{\sigma\nu}\left\{\int_m^{\infty} dp_0 \alpha_{p_0\sigma\nu}\varphi^{(+)}_{p_0\sigma\nu}(z)e^{-ip_0t} + \int_{-\infty}^{-m} dp_0 \beta^{+}_{p_0\sigma\nu}\varphi^{(-)}_{p_0\sigma\nu}(z)e^{-ip_0t}\right\} +$$

$$+\sum_{n\sigma}\alpha_{n\sigma}\varphi^{(+)}_{n\sigma}(z)e^{-ip_{0n}t} \tag{2}$$

In this expression $\varphi^{(\pm)}_{p_0\sigma\nu}(z)$ and $\varphi^{(+)}_{n\sigma}(z)$ are the solution of the Dirac equation in the field (1) ( see appendices ); the signs '+' and '-' correspond to the electron and positron states, respectively; the functions $\varphi^{(\pm)}_{p_0\sigma\nu}$ and $\varphi^{(+)}_{n\sigma}$ describe the continuous spectra states and the bound states of electron in the well, respectively; $p_0$ and $\sigma$ are the energy and the spin quantum number, $\nu(\nu = \pm 1)$ is the quantum number characterising the doubly degenerate states with the fixed values of $p_0$ and $\sigma$; $n$ is the quantum number pertaining to the discrete electron levels with the energy $p_{0n}$ in the well; $\alpha_{p_0\sigma\nu}, \beta^{+}_{p_0\sigma\nu}$ and $\alpha_{n\sigma}$ are the second quantisation operators for the fermi-particles obeying the ordinary anticommutation relations and equalities $\alpha_{p_0\sigma\nu}|0\rangle = \beta_{p_0\sigma\nu}|0\rangle = \alpha_{n\sigma}|0\rangle = 0$ ( $|0\rangle$ is the vacuum ket-vector ).

Let us suppose that at the moment of time $t = 0$ in the rang $z \succ 0$ the electric field with intensity $\mathcal{E}$ is switched on. The total potential energy of the system is given by

$$V(z,t) = V_0(z) - e_0\mathcal{E}z\theta(z)\theta(t), \qquad (e_0\mathcal{E} > 0) \tag{3}$$

The electron-positron field operator in the electric field is now written as follows

$$\Psi(z,t) = \sum_{\sigma\nu}\left\{\int_m^{\infty} dp_0\alpha_{p_0\sigma\nu}\psi^{(+)}_{p_0\sigma\nu}(z,t) + \int_{-\infty}^{-m} dp_0\beta^{+}_{p_0\sigma\nu}\psi^{(-)}_{p_0\sigma\nu}(z,t)\right\} +$$

$$+\sum_{n\sigma}\alpha_{n\sigma}\psi^{(+)}_{n\sigma}(z,t) \tag{4}$$

Here $\psi^{(\pm)}_{p_0\sigma\nu}(z,t)$ and $\psi^{(+)}_{n\sigma}(z,t)$ are the Dirac equation solutions in the field with the potential energy (3) satisfying the initial conditions:

$$\psi^{(\pm)}_{p_0\sigma\nu}(z,0) = \varphi^{(\pm)}_{p_0\sigma\nu}(z), \qquad \psi^{(+)}_{n\sigma}(z,0) = \varphi^{(+)}_{n\sigma}(z) \tag{5}$$

The electric current density in the vacuum state $|0\rangle$ and in the state $\alpha^{+}_{n\sigma}|0\rangle$, corresponding to the bound electron state in the well is defined by the formulae

$$j_v(z,t) = e\langle 0|\Psi^{+}(z,t)\alpha_z\Psi(z,t)|0\rangle$$

$$j_{n\sigma}(z,t) = e\langle 0|\alpha_{n\sigma}\Psi^{+}(z,t)\alpha_z\Psi(z,t)\alpha^{+}_{n\sigma}|0\rangle, \qquad \alpha_z \equiv \gamma_0\gamma_z \tag{6}$$

Substituting the field operator expression (4) into (6) we arrive at the following relationships:

$$j_v(z,t) = e\sum_{\sigma\nu}\int_{-\infty}^{-m} dp_0\left[\psi^{(-)}_{p_0\sigma\nu}(z,t)\right]^{+}\alpha_z\psi^{(-)}_{p_0\sigma\nu}(z,t)$$

$$j_{n\sigma}(z,t) = j_v(z,t) + e\left[\psi^{(+)}_{n\sigma}(z,t)\right]^{+}\alpha_z\psi^{(+)}_{n\sigma}(z,t) \tag{7}$$

With the aid of the formulae in the appendices one can easily prove the equalities

$$\left[\varphi^{(-)}_{p_0\sigma\nu}(z)\right]^{+}\alpha_z\varphi^{(-)}_{p_0\sigma\nu}(z) = -\nu\left(2\pi\left|\beta^{(1)}_1\right|\right)^{-1} \tag{8}$$

$$\left[\varphi^{(+)}_{n\sigma}(z)\right]^{+}\alpha_z\varphi^{(+)}_{n\sigma}(z) = 0$$

It is seen from (7) and (8), that

$$j_v(z,t) = j_{n\sigma}(z,t) = 0 \quad \text{at} \quad t \le 0 \tag{9}$$

The quantities $j_v$ and $j_{n\sigma}$ have the following physical meaning: $j_v$ is the electric current density produced by electron-positron pairs created under the influence of the electric field, $j_{n\sigma}$ is the tunnel current density (the emission current) out of the potential well.

According to (7) the calculation of the electron current density reduces to evaluating the wavefunctions $\psi^{(-)}_{p_0\sigma\nu}(z,t)$ and $\psi^{(+)}_{n\sigma}(z,t)$. We expand these functions in terms of the exact solutions of the Dirac equation in the field

$$V(z) = V_0(z) - e_0\mathcal{E}\, z\theta(z) \tag{10}$$

(see appendices). The expansions mentioned above are of the form ( at $t \geq 0$ )

$$\psi^{(-)}_{p_0\sigma\nu}(z,t) = \int_{-m}^{+m} dp_0' e^{-ip_0't} \sum_{\sigma'} a_{p_0'\sigma'}(p_0\sigma\nu)\phi_{p_0'\sigma'}(z), \tag{11}$$

$$\psi^{(+)}_{n\sigma}(z,t) = \int_{-m}^{+m} dp_0' e^{-ip_0't} \sum_{\sigma'} a_{p_0'\sigma'}(n\sigma)\phi_{p_0'\sigma'}(z), \tag{12}$$

where $a_{p_0'\sigma'}(p_0\sigma\nu)$ and $a_{p_0'\sigma'}(n\sigma)$ are the constant coefficients defined by the initial conditions (5), the functions $\phi_{p_0'\sigma'}(z)$ being determined by the formulae (A2.2). In the expansions (11)-(12) we have retained only the eigenfunctions $\phi_{p_0'\sigma'}(z)$ with the energy $p_0' \in (-m, m)$.Allowance for the eigenfunctions with energies lying outside the above mentioned interval would allow one can to calculate the wavefunction part which describes the spreading out of the wavepacket in time (Drukarjev 1951, Oleinik and Arepjev 1983); in this paper, however, we shall not be interested in the spreading out of the wavepacket in time. In this case the main contribution to the electric current density comes from the poles of integrands in (11)-(12) which occur only at $\operatorname{Re} p_0' \in (-m, m)$ .

## 3. Electron-positron pair creation

Let us consider the wavefunction $\psi^{(-)}_{p_0'\sigma\nu}(z,t)$. The expansion coefficients (11) are expressed by the formulae

$$a_{p_0'\sigma'}(p_0\sigma\nu) = \int_{-\infty}^{+\infty} dz \phi^+_{p_0'\sigma'}(z)\varphi^{(-)}_{p_0\sigma\nu}(z) = \frac{1}{p_0' - p_0}\int_0^{\infty} dz \phi^+_{p_0'\sigma'}(z) H_{\text{int}}(z)\varphi^{(-)}_{p_0\sigma\nu}(z); \tag{13}$$

$$H_{\text{int}} \equiv -e_0\mathcal{E}z$$

Making use of the formulae in the appendices for the wavefunctions $\phi_{p_0'\sigma'}$ and $\varphi^{(-)}_{p\,\sigma\nu}$ the coefficients (13) may be transformed into the form

$$a_{p_0'\sigma'}(p_0\sigma\nu) = -\delta_{\sigma'\sigma}\frac{2m^2\overline{\delta}'^* d_\nu}{p_0' - p_0}\{r_\nu^{(-)}[c_1'^*\Lambda_1^{(-)}(p_0', p_0) + c_2'^*\Lambda_2^{(-)}(p_0', p_0)] +$$

$$+ r_\nu^{(+)}[c_1'^*\Lambda_1^{(+)}(p_0', p_0) + c_2'^*\Lambda_2^{(+)}(p_0', p_0)]\}, \tag{14}$$

where the following notation is used

$$\Lambda_1^{(\pm)}(p_0', p_0) = e_0\mathscr{E}\int_0^\infty dz\left[\mathscr{D}_{i\lambda}(i\eta') + e^{i\pi/4}\frac{p_0 \pm k_1}{\sqrt{2e_0\mathrm{E}}}\mathscr{D}_{i\lambda-1}(i\eta')\right]ze^{\mp ik_1 z} \tag{15}$$

$$\Lambda_2^{(\pm)}(p_0', p_0) = e_0\mathscr{E}\int_0^\infty dz\left[\mathscr{D}_{-i\lambda-1}(\eta') + e^{i\pi/4}\frac{p_0 \pm k_1}{\sqrt{2e_0\mathscr{E}}}\frac{1}{\lambda}\mathscr{D}_{-i\lambda}(\eta')\right]ze^{\mp ik_1 z} \tag{16}$$

$$r_1^{(\mp)} = \beta_{\pm 1}^{(1)} + \gamma_1^*\beta_{\mp 1}^{(-1)};$$

$$\eta' = e^{-i\pi/4}\frac{2}{\sqrt{2e_0\mathrm{E}}}(p_0' + e_0\mathscr{E}z),$$

$$r_{-1}^{(\mp)} = \gamma_2\beta_{\pm 1}^{(1)} + \beta_{\mp 1}^{(-1)};$$

Taking into account the equalities (14) and (A2.2) the wavefunction, $\psi_{p_0\sigma\nu}^{(-)}$ may be written as follows ( at $z > 0$ ):

$$\psi_{p_0\sigma\nu}^{(-)}(z,t) = -(2\pi)^{-1}d_\nu e^{\pi\lambda/2}\int_{-m}^{+m}dp_0'\exp(-ip_0't)(p_0' - p_0)^{-1}\times$$

$$\times\{r_\nu^{(-)}[c_1'^*\Lambda_1^{(-)}(p_0', p_0) + c_2'^*\Lambda_2^{(-)}(p_0', p_0)] + r_\nu^{(+)}[c_1'^*\Lambda_1^{(+)}(p_0', p_0) + c_2'^*\Lambda_2^{(+)}(p_0', p_0)]\}\times$$

$$\times\left[\frac{1}{c_1'^*}u_\sigma'\left(i\frac{d}{dz}\right)\mathscr{D}_{-i\lambda}(\eta') + \frac{\lambda}{c_2'^*}u_\sigma'\left(i\frac{d}{dz}\right)\mathscr{D}_{i\lambda-1}(i\eta')\right] \tag{17}$$

where the prime $(\,'\,)$ means that in the corresponding quantity one ought to replace $p_0$ by $p_0'$ ( for example $c_1' = c_1\big|_{p_0 = p_0'}$ ).

The integrand poles in (17) coincide with zeros of the functions $c_1'^*$ and $c_2'^*$. Taking into account (A2.3), we represent the equalities $c_n^* = 0$ $(n = 1,2)$ in the form

$$\frac{\tilde{\alpha}_1^* + \tilde{\alpha}_{-1}^*}{\tilde{\alpha}_1^* - \tilde{\alpha}_{-1}^*} = \frac{k_2}{V_0 - ia_n^*}; \qquad (n = 1,2) \tag{18}$$

where
$$a_1 = -e^{-i\pi/4}\sqrt{2e_0\mathscr{E}}\frac{d}{d\eta_0}\ln\mathscr{D}_{i\lambda-1}(i\eta_0)$$

$$a_2 = -e^{-i\pi/4}\sqrt{2e_0\mathscr{E}}\frac{d}{d\eta_0}\ln\mathscr{D}_{-i\lambda}(\eta_0) \tag{19}$$

We confine ourselves to investigating the weak electric field, for which the condition

$$\lambda >> 1 \tag{20}$$

is fulfilled. In addition to (20) we shall henceforth assume that

$$\lambda >> \lambda - \tilde{p}_0^2 >> 1, \qquad \lambda^{-1/2}\left(\lambda - \tilde{p}_0^2\right)^{3/2} >> 1, \qquad \tilde{p}_0 \equiv p_0 / \sqrt{2e_0\mathcal{E}} \tag{21}$$

Assuming the inequalities (20)-(21) to be satisfied, we turn to computing the asymptotic formulae for the functions $\mathcal{D}_{-i\lambda}(\eta_0)$ and $\mathcal{D}_{i\lambda-1}(i\eta_0)$. These functions satisfy the equation

$$\frac{d^2\varphi}{dy^2} + \left(\frac{y^2}{4} - b\right)\varphi = 0 \tag{22}$$

where $y = 2\tilde{p}_0, b = \lambda + i/2$. According to Abramowitz and Stegun (1964) the linearly independent solutions of (22) are of the form

$$\varphi^{(\pm)}(\tilde{p}_0) = \frac{1}{\sqrt[4]{\left(b - \tilde{p}_0^2\right)}} \exp\left\{\pm\left[-\frac{\tilde{p}_0}{\sqrt{\left(b - \tilde{p}_0^2\right)}} - b \arcsin\frac{\tilde{p}_0}{\sqrt{b}} + \frac{1}{8}\frac{d_3}{\sqrt{\left(b - \tilde{p}_0^2\right)^3}}\right]\right\},$$

$$d_3 \equiv -\frac{1}{b}\left(\frac{\tilde{p}_0^3}{6} - b\tilde{p}_0\right). \tag{23}$$

The formulae (23) may be derived by the JWKB method. We represent the functions $\mathcal{D}_{-i\lambda}(\eta_0)$ and $\mathcal{D}_{i\lambda-1}(i\eta_0)$ in the form of linear combinations of the functions $\varphi^{(\pm)}$; in particular

$$\mathcal{D}_{-i\lambda}(\eta_0) = \alpha_1\varphi^{(-)}(\tilde{p}_0) + \alpha_2\varphi^{(+)}(\tilde{p}_0) \tag{24}$$

where the constants $\alpha_n\,(n = 1,2)$ may be defined by using the known expression for the parabolic cylinder functions at $\tilde{p}_0 \to 0$. A simple calculation leads to the following formulae

$$\alpha_1 = \frac{\sqrt{\pi}}{2} e^{-\pi\lambda} 2^{-i\lambda/2} \frac{\sqrt[4]{\lambda + i/2}}{\Gamma\left(\frac{i\lambda}{2} + \frac{1}{2}\right)}; \qquad \alpha_2 = 2\alpha_1\left(e^{\pi\lambda} - \tfrac{1}{2}\right). \tag{25}$$

Making use of the presented relationships and retaining throughout only the largest terms, we obtain

$$a_{\frac{1}{2}} = -\sqrt{\frac{e_0\mathcal{E}}{2}} \frac{d}{d\tilde{p}_0}\left\{\ln \varphi^{(+)} \mp e^{-\pi\lambda}\frac{\varphi^{(-)}}{2\varphi^{(+)}}\right\}$$

Then with the aid of the equalities (23) we find:

$$a_{\frac{1}{2}} = \kappa_1 + \Delta a_{\pm}$$

$$\Delta a_{\pm} = -\frac{e_0\mathcal{E}}{2\kappa_1^2}(p_0 - i\kappa_1) \pm i\kappa_1 \exp\left[-\frac{4}{3}\sqrt{\frac{(\lambda - \tilde{p}_0^2)^3}{\lambda}}\right] \tag{26}$$

The substitution of (26) into (18) gives

$$G(p_0) \equiv (k_2^2 - \kappa_1^2 - V_0^2)\sin k_2 L - 2\kappa_1 k_2 \cos k_2 L =$$

$$= (\Delta a_{\pm})^* [k_2 \cos k_2 L + (\kappa_1 - iV_0)\sin k_2 L] \tag{27}$$

If an electric field is absent the last equality goes over into the dispersion equation $G(p_0) = 0$ (A1.8) defining the energy levels of the bound electron states in the well. Denoting the roots of the equation (27) by $p_{0n} = p_{0n}^{(0)} + \Delta p_{0n} - i\Gamma_n^{(\pm)}$, where $p_{0n}^{(0)}$ are the roots of the dispersion equation (A1.8), we obtain the following equation for defining the quantities $\Delta p_{0n}$ and $\Gamma_n^{(\pm)}$:

$$\left.\frac{dG(p_0)}{dp_0}\right|_{p_0 = p_{0n}^{(0)}} (\Delta p_{0n} - i\Gamma_n^{(\pm)}) = (\Delta a_{\pm})^* [k_2 \cos k_2 L + (\kappa_1 - iV_0)\sin k_2 L]_{p_0 = p_{0n}^{(0)}} \tag{28}$$

From (28) we get

$$\Delta p_{0n} = -\frac{1}{8}\frac{e_0\mathcal{E}}{\kappa_1}\frac{k_2^2}{V_0(p_0 + V_0/2)}\left.\left(1 + \frac{1}{2}\kappa_1 L \frac{p_0 + V_0}{p_0 + V_0/2}\right)^{-1}\right|_{p_0 = p_{0n}^{(0)}} \tag{29}$$

$$\Gamma_n^{(\pm)} = \pm \frac{\kappa_1^2 k_2^2}{4mV_0(p_0 + V_0/2)} \exp\left[-\tfrac{4}{3}(\lambda - \tilde{p}_0^2)^{3/2}\lambda^{-1/2}\right]\left.\left(1 + \frac{1}{2}\kappa_1 L \frac{p_0 + V_0}{p_0 + V_0/2}\right)^{-1}\right|_{p_0 = p_{0n}^{(0)}}$$

The quantities $\Delta p_{0n}$ and $\Gamma_n^{(+)}$ represent the shift and the width of energy level of a quasi-stationary state appeared in the electric field. In order to compare these quantities and the analogous ones ( $\Delta E_n$ and $\Gamma_n$ ) of the non-relativistic theory (Oleinik and Arepjev 1983), we pass to the non-relativistic energy reading by putting,

$$p_0 = m - V_0 + E \tag{30}$$

Then $\kappa_1 = \sqrt{2m(V_0 - E)}, \qquad k_2 = \sqrt{2mE}$

$$\sqrt{\frac{(\lambda - \tilde{p}_0^2)^3}{\lambda}} = \frac{\sqrt{2m}}{e_0\mathcal{E}}\sqrt{(V_0 - E)^3} \equiv \tfrac{3}{2}\tilde{\zeta} \tag{31}$$

Using the last formulae and retaining only the largest terms, we obtain (in the ordinary system of units)

$$\Delta p_{0n} = -\sqrt{\frac{2E^2}{m(V_0 - E)}}\frac{\hbar e_0\mathcal{E}}{8V_0}\left.(1 + \tfrac{1}{2}\kappa_1 L)^{-1}\right|_{E = E_n^{(0)}} \tag{32}$$

$$\Gamma_n^{(\pm)} = \pm(V_0 - E)\frac{E}{V_0}\exp\left(-2\tilde{\zeta}\right)\left(1+\tfrac{1}{2}\kappa_1 L\right)^{-1}\Big|_{E=E_n^{(0)}}$$

The quantities $\Delta p_{0n}$ and $\Gamma_n^{(+)}$ coincide exactly with the non-relativistic ones ( $\Delta E_n$ and $\Gamma_n$ ).

At $t - z > 0$ the integration path in (17) may be closed in the lower half-plane of the complex variable $p_0'$. Taking into account that it is only the function $\left(c_1'^*\right)^{-1}$ that has singularities in the lower half-plane of $p_0'$ and retaining in (17) only the singular terms, we obtain

$$\psi_{p_0\sigma\nu}^{(-)}(z,t) = id_\nu e^{\pi\lambda/2}\sum \underset{p_0'=p_{0n}}{\mathrm{Re}\,s}\left\{e^{-ip_0't}\left(p_0' - p_0\right)^{-1}\frac{c_2'^*}{c_1'^*}\times\right. \tag{33}$$

$$\left.\times\left[r_\nu^{(-)}\Lambda_2^{(-)}(p_0', p_0) + r_\nu^{(+)}\Lambda_2^{(+)}(p_0', p_0)\right]u_\sigma'\left(i\frac{d}{dz}\right)\mathcal{D}_{-i\lambda}(\eta')\right\}$$

where the symbol $\underset{p_0'=p_{0n}}{\mathrm{Re}\,s}$ means the residue in the pole defined by the formula

$$p_{0n} = p_{0n}^{(0)} + \Delta p_{0n} - i\Gamma_n^{(+)}$$

For convenience, we represent the expression (33) in the form

$$\psi_{p_0\sigma\nu}^{(-)}(z,t) = \sum_n g_{n\nu}(p_0)e^{-ip_{0n}t}u_{\sigma n}\left(i\frac{d}{dz}\right)\mathcal{D}_{-i\lambda}(\eta_n) \tag{34}$$

$$g_{n\nu}(p_0) = -id_\nu e^{\pi\lambda/2}\sqrt{\lambda}\,e^{i(\lambda\ln\lambda-\lambda-\pi/4)}\left(p_{0n} - p_0\right)^{-1}\left(1+\tfrac{1}{2}\kappa_{1n}L\right)^{-1}\times$$

$$\times\frac{\kappa_{1n}^2 k_{2n}^2}{2m^2V_0}\left[r_\nu^{(-)}\Lambda_2^{(-)}(p_{0n}, p_0) + r_\nu^{(+)}\Lambda_2^{(+)}(p_{0n}, p_0)\right]\exp\left(-2\tilde{\zeta}_n\right) \tag{35}$$

$$\kappa_{1n} = \kappa_1\big|_{p_0=p_{0n}}, \qquad k_{2n} = k_2\big|_{p_0=p_{0n}}, \qquad \tilde{\zeta}_n = \frac{2}{3}\sqrt{\frac{\left(\lambda - \tilde{p}_0^2\right)^3}{\lambda}}\Bigg|_{p_0=p_{0n}}$$

$$u_{\sigma n}\left(i\frac{d}{dz}\right)\mathcal{D}_{-i\lambda}(\eta_n) = u_\sigma\left(i\frac{d}{dz}\right)\mathcal{D}_{-i\lambda}(\eta)\Big|_{\eta=\eta_n}; \qquad \eta_n = \eta\big|_{p_0=p_{0n}}$$

In deriving the relationships (34)-(35) we have used the equality

$$\underset{p_0=p_{0n}}{\mathrm{Re}\,s}\frac{c_2^*}{c_1^*} = -\sqrt{\lambda}e^{i(\lambda\ln\lambda-\lambda-\pi/4)}\frac{\kappa_{1n}^2 k_{2n}^2}{2m^2V_0}\frac{\exp\left(-2\tilde{\zeta}_n\right)}{1+\frac{1}{2}\kappa_{1n}L}$$

Let us calculate the electric current density in the state (34) keeping only one term in the sum over $n$ :

$$\left[\psi_{p_0\sigma\nu}^{(-)}\right]^+\alpha_z\psi_{p_0\sigma\nu}^{(-)} = 2m^2e^{-\pi\lambda/2}\left|g_{n\nu}(p_0)\right|^2 e^{-2\Gamma_n^{(+)}(t-z)} \tag{36}$$

Here the equality

$$\left|\mathscr{D}_{-i\lambda}(\eta_n)\right|^2-\lambda\left|\mathscr{D}_{-i\lambda-1}(\eta_n)\right|^2=e^{-\pi\lambda/2}e^{2\Gamma_n^{(+)}z}\qquad\left(\frac{e_0\mathscr{E}z}{m\sqrt{\lambda}}>>1\right)$$

is taken into account.

Further calculation of the vacuum current density reduces to the calculation of the coefficients $\Lambda_2^{(\pm)}(p_0',p_0)$, defined by (15). With the aid of the asymptotic formulae (23) and (24) we obtain the following representation:

$$\Lambda_2^{(\pm)}(p_0',p_0)=e_0\mathscr{E}e^{i\pi/4}2^{-i\lambda/2}\sqrt{\frac{\pi}{\lambda}}\frac{\left(\lambda+\frac{1}{2}i\right)^{1/4}}{\Gamma\left(\frac{1}{2}(i\lambda+1)\right)}\left(1+\frac{\tilde{p}_0\pm\tilde{k}_1}{\sqrt{\lambda}}\right)I_1 \qquad (37)$$

where

$$I_1=\int_0^\infty dz\frac{z}{\left(\lambda-\tilde{p}_{0z}'^2\right)^{1/4}}\exp\left[-2\int_0^{\tilde{p}_{0z}'}dx\left(\lambda-x^2+\tfrac{1}{2}i\right)^{1/2}\mp ik_1z\right] \qquad (38)$$

$$\tilde{p}_{0z}'=\left(p_0'+e_0\mathscr{E}z\right)\left(2e_0\mathscr{E}\right)^{-1/2};\qquad \tilde{k}_1=k_1\left(2e_0\mathscr{E}\right)^{-1/2}$$

The integration in (38) by parts using the equality

$$e^{G(z)}dz=\left[\frac{dG(z)}{dz}\right]^{-1}de^{G(z)}$$

where

$$G(z)=-2\int_0^{\tilde{p}_{0z}'}dx\left(\lambda-x^2+\tfrac{1}{2}i\right)^{1/2}\mp ik_1z$$

yields

$$\int_0^\infty dz\frac{z\exp G(z)}{\left(\lambda-\tilde{p}_{0z}'^2\right)^{1/4}}=\frac{1}{2e_0\mathscr{E}}\frac{\exp\left[-2\int_0^{\tilde{p}_0'}dx\left(\lambda-x^2+\frac{1}{2}i\right)^{1/2}\right]}{\left(\lambda-\tilde{p}_0'^2\right)^{1/4}\left[\left(\lambda-\tilde{p}_0'^2\right)^{1/2}\pm i\tilde{k}_1\right]^2} \qquad (39)$$

Taking into consideration the equalities (37)-(39) we arrive at the following representation for the coefficients $\Lambda_2^{(\pm)}(p_0',p_0)$:

$$\Lambda_2^{(\pm)}(p_0',p_0)=\frac{1}{2}\sqrt{\frac{\pi}{2}}e^{i\pi/4}2^{-i\lambda/2}e^{-\pi\lambda/2}\Gamma^{-1}\left(1+\tfrac{1}{2}i\lambda\right)\left(1+\frac{\tilde{p}_0\pm\tilde{k}_1}{\sqrt{\lambda}}\right)\times$$

$$\times\frac{\lambda^{1/4}\exp\left[\frac{2}{3}\lambda^{-1/2}\left(\lambda-\tilde{p}_0'^2\right)^{3/2}\right]}{\left(\lambda-\tilde{p}_0'^2\right)^{1/4}\left[\left(\lambda-\tilde{p}_0'^2\right)^{1/2}\pm i\tilde{k}_1\right]^2} \qquad (40)$$

With the aid of relationships (35), (360 and (40) we find

$$j_{p_0n} \equiv \sum_{\sigma,\nu}\left[\psi^{(-)}_{p_0\sigma\nu}\right]^{+}\alpha_z\psi^{(-)}_{p_0\sigma\nu} =$$

$$= \frac{1}{8\lambda^2}\frac{mk_{2n}^4}{V_0^2\kappa_{1n}}\left(\frac{k_1}{p_{0n}+|p_0|}\right)^2\left(\frac{\kappa_{1n}}{p_0^2-p_{0n}^2}\right)^4 e^{-2\tilde{\zeta}_n}\left(1+\tfrac{1}{2}\kappa_{1n}L\right)^{-2}\times$$

$$\times\sum_{\nu=\pm1}|d_\nu|^2\left|r_\nu^{(-)}Q_n^{(-)}+r_\nu^{(+)}Q_n^{(+)}\right|^2\exp\left[-2\Gamma_n^{(+)}(t-z)\right] \qquad (41)$$

Here the following notation is used: $Q_n^{(\pm)} \equiv Q^{(\pm)}\left(p_{0n}, p_0\right)$

$$Q^{(\pm)}(p_0', p_0) = \pm\left(1\mp i\frac{k_1}{\kappa_1'}\right)^2\left(1\mp\frac{k_1}{|p_0|+m}\right)$$

$$\tilde{\zeta}_n = \frac{2}{3}\frac{\sqrt{2m}}{e_0\mathscr{E}}(V_0-E_n)^{3/2}, \qquad E_n = p_{0n}^{(0)} - m + V_0$$

Making use of the equalities (16), (A1.12) and (A1.13), one can readily derive the formula

$$\sum_{\nu=\pm1}|d_\nu|^2\left|r_\nu^{(-)}Q_n^{(-)}+r_\nu^{(+)}Q_n^{(+)}\right|^2 =$$

$$= \frac{|p_0+k_1|}{8\pi m^2k_1}\left\{\left|Q_n^{(-)}\right|^2+\frac{(p_0-k_1)^2}{m^2}\left|Q_n^{(+)}\right|^2+\frac{1}{\left|\beta_1^{(1)}\right|^2}\left[Q_n^{(+)}Q_n^{*(-)}\beta_{-1}^{(1)}\beta_1^{*(1)}+c.c.\right]\right\} \qquad (42)$$

The quantity $j_{p_0n}$ decreases as $p_0^{-6}$ at $p_0^2 >> m^2$.

Let us confine ourselves to the non-relativistic values of the quantity $p_0$ :

$$|p_0| - m \equiv \Delta \le V_0 - E_n$$

In this case the expression in curly brackets in (42) slightly depends on $p_0$ and equals unity in the order of magnitude. To obtain the numerical estimate, we replace this expression by unity. After some simplifications we obtain:

$$j_{p_0n} = \frac{1}{\pi}\left(\frac{E_n}{8\lambda V_0}\right)^2\left(\frac{|p_0|-m}{V_0-E_n}\right)^{\frac{1}{2}}\left(\frac{V_0-E_n}{|p_0|-p_{0n}}\right)^4 e^{-2\tilde{\zeta}_n}\left(1+\tfrac{1}{2}\kappa_{1n}L\right)^{-2}\exp\left[-2\Gamma_n^{(+)}(t-z)\right] \qquad (43)$$

According to (43) the dependence of the quantity $j_{p_0n}$ on the field is of the form:

$$j_{p_0 n} \approx (e_0 \mathcal{E})^2 \Gamma_n^{(+)}$$

The appearance of the factor $\exp\left[-2\Gamma_n^{(+)}(t-z)\right]$ in (43) allows one to interpret the quantity $j_{p_0 n} dp_0$ in the following way. This quantity is a flux of electrons created in pairs together with positrons in the well under the influence of an electric field $\mathcal{E}$ and moving away from the barrier in the direction $z \to +\infty$ at the velocity of light, the energy of positrons formed in the field lying in the range $\left(-p_0, -p_0 - dp_0\right)$. It is obvious that the quantity $-e_0 \int_{-\infty}^{-m} j_{p_0 n} dp_0 \equiv Q$ at $t - z \geq 0$, $z \to +\infty$ is the total electric charge of electrons created per unit time in the field under consideration. The peculiar feature of the given model consists in the fact that in the range $z \to -\infty$ the positron flux does not occur; the total electric charge of positrons equal to $-Q$ is concentrated near the boundary $z = -L$ of the potential well.

Let us estimate the quantity

$$j_{V_n} \equiv \int_{-(m+\Delta)}^{-m} dp_0\, j_{p_0 n} ,$$

assuming that the condition $\Delta << V_0 - E_n$ is satisfied. We may approximately put:

$$\int_{-(m+\Delta)}^{-m} dp_0 \left( \frac{|p_0| - m}{V_0 - E_n} \right)^{\frac{1}{2}} \left( \frac{V_0 - E_n}{|p_0| - p_{0n}} \right)^4 \approx \tfrac{2}{3} \Delta^{3/2} \left(V_0 - E_n\right)^{-1/2}$$

Estimate the vacuum current for the following values of parameters of the problem

$$\mathcal{E} = 5 \cdot 10^6 \; v/cm, \qquad V_0 = 1ev, \qquad E_n / V_0 = \tfrac{1}{2}, \qquad \Delta / V_0 - E_n = \tfrac{1}{5} \tag{44}$$

Write out some auxiliary quantities

$$b \equiv V_0 / e_0 \mathcal{E} = 0.2 \cdot 10^{-6}\, cm; \qquad \lambda = m^2 c^3 / 2\hbar e_0 \mathcal{E} = 1.3 \cdot 10^9;$$

$$\lambda - \tilde{p}_{0n}^2 = 0.3 \cdot 10^4; \qquad 2\tilde{\zeta}_n = 4.8$$

Putting additionally $\kappa_1 L \approx 1$, we obtain

$$j_{V_n} \approx 10^{-10}\, sec^{-1} \tag{45}$$

It should be emphasised that the pair creation probability in electric field in the presence of the potential well is by no means exponentially small. Remember that according to Schwinger (Schwinger 1951) the pair creation probability in an electric field in the absence of the well is proportional to the exponent $e^{-2\pi\lambda}$ which equals $\exp\left(-8\cdot 10^{9}\right)$ for the chosen parameters (44). Compare this quantity with the exponent $\exp\left(-2\tilde{\zeta}_n\right)$ involved in (43): for the same parameters the latter exponent is equal to $\exp(-4.8)$ !

# 4. Electron emission

Now we turn to calculating the tunnel current of electron emission defined by the wavefunction $\psi^{(+)}_{n\sigma}(z,t)$ (12). The coefficient $a_{p_0'\sigma'}(n\sigma)$ in (12) is expressed by

$$a_{p_0'\sigma'}(n\sigma)=\frac{1}{p_0'-p_{0n}^{(0)}}\int_0^\infty dz\phi^{+}_{p_0'\sigma'}(z)H_{\text{int}}(z)\varphi^{(+)}_{n\sigma}(z)$$

Using the formulae in the appendix we obtain the following representation

$$a_{p_0'\sigma'}(n\sigma)=-2m^2\bar{\delta}^*_{p_0'}\tilde{\delta}_n\left(\tilde{\alpha}_{1n}+\tilde{\alpha}_{-1n}\right)\left(p_0'-p_{0n}^{(0)}\right)^{-1}\times$$

$$\times\left[c_1^*(p_0')\Lambda_1\left(p_0',p_{0n}^{(0)}\right)+c_2^*(p_0')\Lambda_2\left(p_0',p_{0n}^{(0)}\right)\right]\delta_{\sigma'\sigma} \qquad (46)$$

$$\Lambda_1\left(p_0',p_{0n}^{(0)}\right)=e_0\mathcal{E}\int_0^\infty dz\left[\mathcal{D}_{i\lambda}(i\eta')+\left(\tilde{p}_{0n}^{(0)}-i\tilde{\kappa}_{1n}\right)e^{i\pi/4}\mathcal{D}_{i\lambda-1}(i\eta')\right]ze^{-\kappa_{1n}z}$$

$$\Lambda_2\left(p_0',p_{0n}^{(0)}\right)=e_0\mathcal{E}\int_0^\infty dz\left[\mathcal{D}_{-i\lambda-1}(\eta')+\left(\tilde{p}_{0n}^{(0)}-i\tilde{\kappa}_{1n}\right)\tfrac{1}{\lambda}e^{i\pi/4}\mathcal{D}_{-i\lambda}(\eta')\right]ze^{-\kappa_{1n}z} \qquad (47)$$

where $$\tilde{p}_{0n}^{(0)}=\frac{p_{0n}^{(0)}}{\sqrt{2e_0\mathcal{E}}},\qquad \tilde{\kappa}_{1n}=\left.\frac{\kappa_1}{\sqrt{2e_0\mathcal{E}}}\right|_{p_0=p_{0n}^{(0)}},\qquad \eta'=e^{-i\pi/4}\frac{2}{\sqrt{2e_0\mathcal{E}}}\left(p_0'+e_0\mathcal{E}z\right)$$

The formulae (46) and (47) are analogous to ( 14 ) and (15). As is seen from (47) and (15) the coefficient $\Lambda_m\left(p_0',p_{0n}^{(0)}\right)$ may be obtained by the substitution $\mp ik_1\to-\kappa_{1n}$, $p_0\to p_{0n}^{(0)}$ made in the formula for $\Lambda_m^{(\pm)}\left(p_0',p_0\right)$.

Using the formulae presented above and performing the calculations in the same way as in the preceding section, we obtain the following expression for the wavefunction ( at $t-z,\ z\geq 0$ )

$$\psi^{(+)}_{n\sigma}(z,t)=\sum_{\overline{m}}g_{\overline{m}}(n)e^{-ip_{0\overline{m}}t}u_{\sigma\overline{m}}\left(i\frac{d}{dz}\right)\mathcal{D}_{-i\lambda}(\eta_{\overline{m}});\qquad \eta_{\overline{m}}=\eta\big|_{p_0=p_{0\overline{m}}}$$

$$g_{\overline{m}}(n)=-i\tilde{\delta}_n\left(\tilde{\alpha}_{1n}+\tilde{\alpha}_{-1n}\right)e^{\pi\lambda/2}\sqrt{\lambda}\exp\left[i(\lambda\ln\lambda-\lambda-\pi/4)\right]\times$$

$$\times\left(p_{0\overline{m}}-p_{0n}^{(0)}\right)^{-1}\left(1+\tfrac{1}{2}\kappa_{1\overline{m}}L\right)^{-1}\frac{\kappa_{1\overline{m}}^{2}k_{2\overline{m}}^{2}}{2m^{2}V_{0}}\Lambda_{2}\left(p_{0\overline{m}},p_{0n}^{(0)}\right)\exp\left(-2\tilde{\zeta}_{\overline{m}}\right) \tag{48}$$

Calculate the electron current density in the state $\psi_{n\sigma}^{(+)}(z,t)$ keeping only one term in the sum over $\overline{m}$ in (48)

$$j_{n}\equiv\left[\psi_{n\sigma}^{(+)}\right]^{+}\alpha_{z}\psi_{n\sigma}^{(+)}=2m^{2}e^{-\pi\lambda/2}\left|g_{\overline{m}}(n)\right|^{2}e^{-2\Gamma_{\overline{m}}^{(+)}(t-z)} \tag{49}$$

Further, take into account the equality

$$\left|\tilde{\alpha}_{1}+\tilde{\alpha}_{-1}\right|^{2}=e^{-2\kappa_{1}L} \qquad (\text{ at } p_{0}=p_{0n}^{(0)}),$$

and make use of the formula for the normalisation constant $\tilde{\delta}_{n}$ ( see appendices ) and for the function $\Lambda_{2}^{(\pm)}\left(p_{0}',p_{0}\right)$ (40). The final result is:

$$\begin{aligned} j_{n}=&\left[2m\left(V_{0}-E_{n}\right)\right]^{1/2}\frac{E_{n}}{V_{0}}\left(\frac{E_{\overline{m}}}{V_{0}}\right)^{2}\left(\frac{V_{0}-E_{\overline{m}}}{p_{0\overline{m}}-p_{0n}^{(0)}}\right)^{2}\sqrt{\lambda}\left(1+\tfrac{1}{2}\kappa_{1n}L\right)^{-1}\times \\ &\times\left(1+\tfrac{1}{2}\kappa_{1\overline{m}}L\right)^{-2}\frac{\exp\left[-\frac{4}{3}\left(\lambda-\tilde{p}_{0\overline{m}}^{2}\right)^{3/2}\lambda^{-1/2}\right]}{\left(\lambda-\tilde{p}_{0\overline{m}}^{2}\right)^{1/2}\left[\left(\lambda-\tilde{p}_{0\overline{m}}^{2}\right)^{1/2}+\tilde{\kappa}_{1n}\right]^{4}}\exp\left[-2\Gamma_{\overline{m}}^{(+)}(t-z)\right] \end{aligned} \tag{50}$$

Consider the case of the one-level potential well. Assuming $\overline{m}=n$ and using formula (34) for the energy level shift, we arrive at the following expression for the tunnel current

$$j_{n}\Big|_{t-z=0}=2E_{n}\frac{V_{0}-E_{n}}{V_{0}}\left(1+\tfrac{1}{2}\kappa_{1n}L\right)^{-1}\exp\left[-\frac{4}{3\sqrt{\lambda}}\left(\lambda-\tilde{p}_{0n}^{2}\right)^{3/2}\right]=2\Gamma_{n}^{(+)} \tag{51}$$

For the chosen values of parameters (44), we obtain

$$j_{n}\approx 4\cdot 10^{12}\,s^{-1}$$

Note that expression (51) coincides exactly with the formula for tunnel current derived within the stationary non-relativistic theory, which corresponds to the adiabatic switching on of an electric field. At the same time the tunnel current (51) is much greater in magnitude than the emission current being predicted by the non-relativistic theory in the case of sudden switching on of an electric field (Oleinik and Arepjev 1983). Thus, the tunnel current value ( at sudden switching on of the field ) is affected appreciably by allowance for the positron band.

According to the relationships (43) and (51) the electric current densities in the vacuum state and in the one-electron state corresponding to the discrete level in the well contain one and the same exponential factor. The difference between their pre-exponential factors is, however, very considerable. This is due to the fact that in the present problem the electron-positron pair creation is a two-step process: first, under the influence of an electric field the negative-frequency electrons go over to the discrete levels in the well with the probability proportional to $\mathcal{E}^2$ and then the tunnel passage of electrons out of the well through the barrier formed by an electric field takes place.

The electron-positron pair creation mechanism investigated in the paper is due to the appearance of the non-zero width of energy levels of elementary excitations. In an electric field the electron-positron vacuum is polarised and transformed into the unstable medium, the elementary excitations of which are disintegrated and owing to this induce the tunnel electric current.

As is known (Landau and Lifshitz 1958, Blokhintsev 1961) the atom ionisation in a steady electric field is the tunnel electron transition through the potential barrier formed by the field. The tunnelling in an electric field seems to be a phenomenon which may be accurately enough described by means of the one-dimensional potential well model. Therefore, the results presented here allow us to think that the atom ionisation in an electric field is accompanied by processes of the electron-positron pair creation, which may be quite well observed at relatively weak fields. Really, assuming the pair creation probability in the hydrogen atom field and in an electric field to be equal to (45), we find out that each second in $1\,cm^3$ of the hydrogen gas taken at normal conditions the pairs are created with the total energy of the order of $10^{-4}\,J$ .

As was explained above, the distinctive feature of the model considered in the paper is the absence at $z \to -\infty$ of the flux of positrons created in the well. It is due to the fact that the applied field $\mathcal{E}$ is different from zero only at $z \succ 0$ . One may show that if the field $\mathcal{E}$ is non-vanishing at all values of $z$ there will be a potential barrier at $z \prec 0$ and, consequently, a flux of positrons will occur at $z \to -\infty$ . Positrons penetrating the barrier may reach the plate of a capacitor and annihilate with electrons of the plate, producing destructions (microexplosions) in it. This process of annihilation need not be accompanied by the X-ray emission.

**Acknowledgments**

We are grateful to G F Drukarjev, S A Moskalenko and A A Borkhardt for interest in the research, helpful discussions and stimulating support.

## Appendix 1. Wavefunctions of the relativistic electron in a potential well and in an electric field

We seek the solution of the Dirac equation

$$\left(i\hat{\partial} - \gamma_0 V - m\right)\varphi(z,t) = 0 \qquad \text{(A1.1)}$$

in the field with the potential energy $V = V(z)$ in the form

$$\varphi(z,t) = \exp\left(-ip_0 t + ip_x x + ip_y y\right) u_\sigma(id/dz)\tilde{\varphi}(z), \; (\sigma = \pm 1). \qquad \text{(A1.2)}$$

Here $\hat{¶} = \gamma_0 \frac{\partial}{\partial \cdot t} + \mathbf{g}\frac{\partial}{\partial \cdot \mathbf{r}}$, $\qquad p_\pm = p_x \pm ip_y$

$$u_1\left(i\frac{d}{dz}\right) = \begin{pmatrix} p_0 - V + m + i\,d/dz \\ -p_+ \\ -p_0 + V + m - i\,d/dz \\ p_+ \end{pmatrix}; \; u_{-1}\left(i\frac{d}{dz}\right) = \begin{pmatrix} p_- \\ p_0 - V + m + i\,d/dz \\ p_- \\ p_0 - V - m + i\,d/dz \end{pmatrix} \qquad \text{(A1.3)}$$

The function $\tilde{\varphi}(z)$ satisfies the equation

$$\left[d^2/dz^2 + i\,dV/dz + (p_0 - V)^2 - m^2 - p_x^2 - p_y^2\right]\tilde{\varphi}(z) = 0. \qquad \text{(A1.4)}$$

*The potential well*. The solution of the equation (A1.4) in the field

$$V = -V_0\theta(z+L)\theta(-z) \qquad (V_0 = const) \qquad \text{(A1.5)}$$

must obey the following continuity conditions

$$\tilde{\varphi}(z)\big|_{z=-L+0} = \tilde{\varphi}(z)\big|_{z=-L-0}$$

$$\left.\frac{d\tilde{\varphi}(z)}{dz}\right|_{z=-L+0} = \left.\frac{d\tilde{\varphi}(z)}{dz}\right|_{z=-L-0} + V_0\tilde{\varphi}(-L) \qquad \text{(A1.6)}$$

$$\tilde{\varphi}(z)\big|_{z=+0} = \tilde{\varphi}(z)\big|_{z=-0}$$

$$\left.\frac{d\tilde{\varphi}(z)}{dz}\right|_{z=+0} = \left.\frac{d\tilde{\varphi}(z)}{dz}\right|_{z=-0} - iV_0\tilde{\varphi}(0)$$

These conditions may be derived most simply by integrating the equation (A1.4) over two ranges lying in the vicinity of the points $z = -L$ and $z = 0$.

The wavefunctions describing the electron bound states in the field (A1.5) are of the form

$\left(p_0^2 \prec m^2 + p_\perp^2,\, p_\perp^2 = p_x^2 + p_y^2\right)$:

$$\varphi_{n\sigma}^{(+)}(\mathbf{r},t) = \varphi_{n\sigma}^{(+)}(z)\exp\left(-ip_0 t + ip_x x + ip_y y\right)$$

$$\varphi_{n\sigma}^{(+)}(z) = \tilde{\delta}_n u_{n\sigma}(i\,d/dz)\left\{\theta(-z-L)e^{\kappa_{1n}z} + \theta(z+L)\theta(-z)\right. \tag{A1.7}$$

$$\left.\times\left(\tilde{\alpha}_{1n}e^{ik_{2n}z} + \tilde{\alpha}_{-1n}e^{-ik_{2n}z}\right) + \theta(z)\left(\tilde{\alpha}_{1n} + \tilde{\alpha}_{-1n}\right)e^{-\kappa_{1n}z}\right\}$$

where

$$\tilde{\delta}_n = \tilde{\delta}\Big|_{p_0 = p_{0n}^{(0)}}; \qquad u_{n\sigma}(i\,d/dz) = u_\sigma(i\,d/dz)\Big|_{p_0 = p_{0n}^{(0)}} \qquad \text{etc.}$$

$$\left|\tilde{\delta}\right|^2 = \frac{1}{4}\kappa_1 k_2^2 m^{-2} V_0^{-1}(2p_0 + V_0)^{-1}\left[1 + \kappa_1 L(p_0 + V_0)/(2p_0 + V_0)\right]^{-1}\exp 2\kappa_1 L$$

$$\tilde{\alpha}_\sigma = \frac{1}{2}\left[1 + \sigma(V_0 - i\kappa_1)/k_2\right]\exp(i\sigma k_2 L - \kappa_1 L)$$

$$\kappa_1 = \left(m^2 + p_\perp^2 - p_0^2\right)^{1/2}, \qquad k_2 = \left[(p_0 + V_0)^2 - m^2 - p_\perp^2\right]^{1/2}.$$

The wavefunctions $\varphi_{n\sigma}^{(+)}$ obey the orthonormalisation condition

$$\int dz\left[\varphi_{n\sigma}^{(+)}(z)\right]^+ \varphi_{n'\sigma'}^{(+)}(z) = \delta_{\sigma'\sigma}\delta_{n'n}$$

The quantities $p_{0n}^{(0)}$, representing the energy levels of electron bound states, are the roots of the dispersion equation

$$\left(k_2^2 - \kappa_1^2 - V_0^2\right)\sin k_2 L - 2\kappa_1 k_2 \cos k_2 L = 0 \tag{A1.8}$$

For simplicity, we confine ourselves to considering the shallow potential well $V_0 \prec\prec m$. In this case the equation (A1.8) has the roots only if the inequalities $(p_0 + V_0)^2 - m^2 - p_\perp^2 > 0$, $p_0^2 - m^2 - p_\perp^2 < 0$ are fulfilled. Note that these inequalities can be carried out simultaneously only for electron states, i. e. at $p_0 > 0$.

We shall from now on put $p_x = p_y = 0$, confining ourselves to the one-dimensional problem with the variable $z$.

The wavefunctions of the continuous spectra states $\left(p_0^2 \ge m^2\right)$ are defined by

$$\varphi_{p_0\sigma\nu}^{(\pm)}(z,t) = \exp(-ip_0 t)\varphi_{p_0\sigma\nu}^{(\pm)}(z) \tag{A1.9}$$

$$\varphi_{p_0\sigma\nu}^{(\pm)}(z) = d_\nu u_\sigma \begin{cases} \varphi_1^{(\pm)}(z) + \gamma_1^*\varphi_{-1}^{(\pm)}(z) & at \quad \nu = 1 \\ \gamma_2\varphi_1^{(\pm)}(z) + \varphi_{-1}^{(\pm)}(z) & at \quad \nu = -1 \end{cases}$$

where the '+' and'-' signs correspond to the electron $\left(p_0 \geq m\right)$, and positron $\left(p_0 \leq -m\right)$, states, $d_n$ is the normalisation constant, the constants $\gamma_1$ and $\gamma_2$ are defined by the equalities

$$\left[\varphi^{(\pm)}_{p_0\sigma\nu}\right]^{+} \alpha_z \varphi^{(\pm)}_{p_0\sigma-\nu} = 0 \qquad \text{(A1.10)}$$

$$\int dz \left[\varphi^{(\pm)}_{p_0'\sigma'\nu'}(z)\right]^{+} \varphi^{(\pm)}_{p_o\sigma\nu}(z) = \delta_{\sigma'\sigma}\delta_{\nu'\nu}\delta\left(p_0' - p_0\right)$$

The function $\varphi^{(\pm)}_{\nu}(z)$ are of the form:

$$\varphi^{(\pm)}_{\nu}(z) = \theta(-z-L)\exp(i\nu k_1 z) + \theta(z+L)\theta(-z)\left(\alpha^{(\nu)}_{1}\exp(i\nu k_2 z) + \alpha^{(\nu)}_{-1}\exp(-i\nu k_2 z)\right)$$

$$+\,\theta(z)\left(\beta^{(\nu)}_{1}\exp(i\nu k_1 z) + \beta^{(\nu)}_{-1}\exp(-i\nu k_1 z)\right); \qquad \nu = \pm 1$$

$$k_1 = \left(p_0^2 - m^2\right)^{1/2}, \qquad \text{(A1.11)}$$

$$\alpha^{(\nu)}_{\pm 1} = \tfrac{1}{2}\left[1 \pm (\nu k_1 + V_0)/\nu k_2\right]\exp(\pm i\nu k_2 L - i\nu k_1 L);$$

$$\beta^{(\nu)}_{\pm 1} = \tfrac{1}{2}\left[1 \pm (\nu k_2 - V_0)/\nu k_1\right]\alpha^{(\nu)}_{1} + \tfrac{1}{2}\left[1 \mp (\nu k_2 + V_0)/\nu k_1\right]\alpha^{(\nu)}_{-1}$$

Some relationships for the coefficients $\beta^{(\nu)}_{\sigma}$ used in this paper are given:

$$\left|\beta^{(\nu)}_{-1}\right|^2 = \left(1 - \left|\beta^{(\nu)}_{1}\right|^2\right)(k_1 - \nu p_0)/(k_1 + \nu p_0),$$

$$\beta^{*(\nu)}_{1}\beta^{*(-\nu)}_{-1} + \beta^{*(\nu)}_{-1}\beta^{(-\nu)}_{1}\,(k_1 + \nu p_0)/(k_1 - \nu p_0) = 0, \qquad \text{(A1.12)}$$

$$\left|\beta^{(\nu)}_{1}\right|^2 = \left|\beta^{(-\nu)}_{1}\right|^2, \qquad \left|\beta^{(-1)}_{-1}\right|^2 = \left|\beta^{(1)}_{-1}\right|^2\left[(k_1 + p_0)/(k_1 - p_0)\right]^2.$$

The constants $\gamma_1, \gamma_2$ and $d_\nu$ are expressed by

$$\gamma_1 = \left|\beta^{(1)}_{1}\right|\left(1 - \left|\beta^{(1)}_{1}\right|\right)\Big/\beta^{(1)}_{1}\beta^{*(-1)}_{-1}, \qquad \gamma_2 = \gamma_1 (p_0 + k_1)/(p_0 - k_1),$$

$$\left|d_\nu\right|^2 = \left(1 + \left|\beta^{(1)}_{1}\right|\right)\Big/16\pi\cdot\left|\beta^{(1)}_{1}\right|\left|k_1\right|\left|p_0 - \nu k_1\right|. \qquad \text{(A1.13)}$$

## Appendix 2. The potential well and the steady electric field

We present the solution of equation (A1.4) in the field

$$V = -V_0\theta(z+L)\theta(-z) - e_0\mathscr{E}z\theta(z) \qquad \text{(A2.1)}$$

at $p_0^2 \leq m^2$. Taking into account that the continuity conditions (A1.6) are also valid for the field (A2.1) we obtain the following formulae for the wave electron function

$$\phi_{p_0\sigma}(z,t) = e^{-ip_0 t}\phi_{p_0\sigma}(z) \qquad \text{(A2.2)}$$

$$\phi_{p_0\sigma}(z) = \bar{\delta} u_\sigma(id/dz)\{\theta(-z-L)e^{\kappa_1 z} + \theta(z+L)\theta(-z)(\tilde{\alpha}_1 e^{ik_2 z} + \tilde{\alpha}_{-1}e^{-ik_2 z})$$

$$+\theta(z)[c_1\mathscr{D}_{-i\lambda}(\eta) + c_2\mathscr{D}_{i\lambda-1}(i\eta)]\}$$

where the quantities $\kappa_1, k_2$ and $\tilde{\alpha}_\sigma$ are defined by the equalities (A1.7) and the rest of the notation has the following meaning: $\mathscr{D}_{-i\lambda}(\eta)$ and $\mathscr{D}_{i\lambda-1}(i\eta)$ are the parabolic cylinder functions,

$$\lambda = m^2/2e_0\mathscr{E}, \qquad \eta = e^{-i\pi/4}\zeta, \qquad \zeta = 2(2e_0\mathscr{E})^{-1/2}(p_0 + e_0\mathscr{E}z) \qquad \text{(A2.3)}$$

$$c_1 = e^{\pi\lambda/2}\left\{i(\tilde{\alpha}_1 + \tilde{\alpha}_{-1})\frac{d}{d\eta_9}\mathscr{D}_{i\lambda-1}(i\eta_0) + e^{i\pi/4}(2e_0\mathscr{E})^{-1/2}[k_2(\tilde{\alpha}_1 - \tilde{\alpha}_{-1}) - V_0(\tilde{\alpha}_1 + \tilde{\alpha}_{-1})]\mathscr{D}_{i\lambda-1}(i\eta_0)\right\}$$

$$c_2 = -e^{\pi\lambda/2}\left\{i(\tilde{\alpha}_1 + \tilde{\alpha}_{-1})\frac{d}{d\eta_0}\mathscr{D}_{-i\lambda}(\eta_0) + e^{i\pi/4}(2e_0\mathscr{E})^{-1/2}[k_2(\tilde{\alpha}_1 - \tilde{\alpha}_{-1}) - V_0(\tilde{\alpha}_1 + \tilde{\alpha}_{-1})]\mathscr{D}_{-i\lambda}(\eta_0)\right\}$$

$$\eta_0 = \eta|_{z=0}, \qquad |\bar{\delta}|^2 = e^{\pi\lambda/2}\left(4\pi|c_1|^2 m^2\right)^{-1}.$$

The constants $c_1$ and $c_2$ are related by the equality ( at $p_0 = \mathrm{Re}\, p_0$ )

$$\lambda\cdot|c_1|^2 = |c_2|^2 \qquad \text{(A2.4)}$$

The following normalisation conditions are carried out

$$\int dz \phi^+_{p_0'\sigma'}(z)\phi_{p_0\sigma}(z) = \delta_{\sigma\sigma'}\delta(p_0' - p_0) \qquad \text{(A2.5)}$$

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