# Geometric and Topological Structures Related to Universal Algebras

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**Abstract.** In the paper, a special class of universal algebras is introduced (the so-called tame $A$-systems) that includes free algebras with an arbitrary set of operations, their commutative and idempotent modifications, and some other objects. It turns out that there is a system of subsets (the so-called planar subsets) of the support of an arbitrary tame $A$-system, and this system is closed with respect to the operations and forms a semimodular lattice similar to the system of subspaces of a projective space. This structure enables one to extend the metric and topology from the system of generators of the $A$-system to the entire $A$-system in such a way that the Archimedean and Hausdorff properties are preserved under the extension.

## INTRODUCTION

**1.** The paper is devoted to universal algebras. Recall that a *universal algebra* is defined by a set $U$ with an arbitrarily given collection $F$ of mappings $f\colon U^n \to U$, $n = 0, 1, 2, \dots$, where the number $n$ depends on $f \in F$. The set $U$ is referred to as the *support* of the universal algebra and the mappings $f\colon U^n \to U$ as *n-ary operations* on $U$, see [1, 2].

The universal algebra with support $U$ and the set of operations $F$ is denoted in the paper by $(U, F)$, the subset of $n$-ary operations of $U$ by $F_n$, and the image of a sequence $(u_1, \dots, u_n) \in U^n$ under an $n$-ary operation $f$ by $f(u_1, \dots, u_n)$. It is assumed that $F$ contains no 0-ary operations $\varnothing \to U$ (under a 0-ary operation, some element $u \in U$ becomes fixed). For brevity, instead of the term "universal algebra" we always use the term "$A$-system."

The theory of universal algebras has been developing extensively since the forties of the last century. Since the notion of universal algebra is sometimes too general, the investigations in this area are related to some restrictions concerning the operations in $F$.

**2.** In the paper, we consider $A$-systems (universal algebras) $(U, F)$ satisfying the following two conditions.

**Condition 1** (uniqueness of decompositions). For any element $u \in U$, there exists at most one representation (up to the order of elements $u_1, \dots, u_n$) of the form

$$u = f(u_1, \dots, u_n), \qquad f \in F, \tag{1}$$

where $u_i \neq u$ for at least one index $i$.

**Condition 2.** The support $U$ is generated by the subset of elements in $U$ which are not representable in the form (1). This means that every element $u \in U$ can be obtained from these elements by a composition of finitely many operations $f \in F$.

We refer to the universal algebras of this kind as *tame $A$-systems.*

The class of tame $A$-systems includes all free, free commutative, and free idempotent $A$-systems. An $A$-system $(U, F)$ is said to be *free* if

(1) the relation
$$f(u_1, \dots, u_m) = f'(u'_1, \dots, u'_n) \tag{2}$$
holds if and only if $f = f'$ (and hence $m = n$) and $u_i = u'_i$ for each $i$;

(2) there are no relations of the form $u = f(u, \dots, u)$, and

(3) the support $U$ is generated by the subset of elements in $U$ not representable in the form $u = f(u_1, \dots, u_n)$.

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The simplest examples of free $A$-systems are as follows: 1) a sequence $\{x_1, \dots, x_n, \dots\}$ with the only unary operation $f(x_n) = x_{n+1}$; 2) $U$ is the set of "words" in the one-letter alphabet: $a$, $aa$, $a(aa)$, $(aa)a$, $(aa)(aa)$ etc.; the only binary operation given on $U$ is the concatenation $f(u,v) = uv$ (juxtaposition of words).

An $A$-system $(U, F)$ is said to be *free commutative* if Condition 1 of the previous definition is replaced by a weaker one, namely, relation (2) holds if and only if $f = f'$ and the sequences $(u_1, \dots, u_m)$ and $(u'_1, \dots, u'_n)$ differ in their order only.

An $A$-system $(U, F)$ is said to be *free idempotent* if

$$u = f(u, \dots, u) \quad \text{for all} \quad u \in U \quad \text{and} \quad f \in F,$$

and there are no other relations.

**3.** The tame $A$-systems $(U, F)$ are endowed with several important structures. These are, first, integral-valued characteristics of elements of the support, namely, the *height* of an element (the number of successive iterations of operations used to construct the given element from the indecomposable ones) and the *length* of an element $u$ (the number of symbols of elements in a "word" expressing $u$ in terms of indecomposable elements). We stress that, in a tame $A$-system, the length and the height of an element are defined uniquely.

Further, to any element $u$ of a tame $A$-system one can assign a finite directed graph of tree type (decomposition scheme) whose vertices are operations used when constructing this element from indecomposable elements.

The support $U$ of any tame $A$-system is naturally endowed with a partial order relation.

Finally, to any tame $A$-system, one can assign a directed graph whose vertices are elements of the support of this system. For any free $A$-system, this graph completely reconstructs the $A$-system up to isomorphism.

These structures are studied in Section 2.

**4.** In Section 3 we assign to any tame $A$-system $(U, F)$ an analog of projective space. We consider the family $L$ of subsets $V \subset U$ that are closed with respect to the operations in $F$ and have the finite base property. The number of base elements is called the *rank* of the set $V$ and is denoted by $r(V)$. Note that an inclusion $V_1 \subset V$ does not imply the inequality $r(V_1) \leqslant r(V)$ because the set $V$ can contain subsets of arbitrarily large rank.

The set $L$ is endowed with the structure of lattice with respect to the operation of intersection. However, this lattice is not semimodular, and therefore cannot be viewed as a geometric object [4, 5].

In the lattice $L$, we choose a family $L_0 \subset L$ of elements $V \in L$ satisfying the following *maximality condition*: there exist no subsets $V' \in L$ strictly containing $V$ and such that $r(V') \leqslant r(V)$; these subsets are said to be *planar*.

We prove that the family $L_0$ of planar subsets is closed with respect to the operation of intersection, and therefore the sets $L_0$ and $L$ are simultaneously endowed with the structure of a lattice with respect to this operation. Note that the operations of union in $L_0$ and $L$ are different, and hence $L_0$ is not a sublattice of $L$.

It turns out that the lattice $L_0$ is semimodular. Thus, one can naturally interpret the elements of $V$ of rank $r$ as planes of dimension $r-1$ in the "projective" (generally infinite-dimensional) space $U$ because these objects satisfy the main axioms on the intersections and unions of planes.

The specific feature of the geometry thus defined is that, on every plane of dimension $n$, we have a uniquely chosen set of $n+1$ points generating the plane (elements of the base). Here the base of the intersection of two planes is contained in the union of the bases of these planes.

Starting from the notion of a planar subset, we then introduce the notion of plane in the support $U$ of a tame $A$-system. An $F$-subset $V \subset U$ of a tame $A$-system $A = (U, F)$ is called a *plane* if every planar subset in $V$ is a planar subset in $U$. In particular, all planar subsets in $U$ are planes.

We prove that each intersection of planes is also a plane. Thus, the set of planes is endowed with the structure of a lattice, and the family of planar subsets forms a sublattice of this lattice.

It is shown that the lattice of planes is also semimodular.

**5.** Sections 4 and 5 are devoted to topological and metric universal $A$-systems. An $A$-system $(U, F)$ is said to be *topological* (*metric*) if the support $U$ of this system is endowed with the structure of a topological (metric) space with respect to which all operations $f \in F$ are continuous.

If $(U, F)$ is a tame $A$-system, then, starting from a topology (metric) on $U$, one can define the topology (metric) on the set $L$ of all subsets $V \subset U$ of finite rank that are closed with respect to the operations in $F$.

Here we restrict ourselves to free, free commutative, and free idempotent $A$-systems only. We construct an extension of the Hausdorff topology and metric, Archimedean and non-Archimedean, which are initially given on the base $X \subset U$ only, to the Hausdorff topology (Archimedean and non-Archimedean metric, respectively) on the entire support $U$.

In the case of free idempotent $A$-system, we construct an extension of the topology to $X$ to another topology on $U$, which is weaker than the previous one. In this new ("secondary") topology, bases of neighborhoods are subsets of $U$ closed with respect to the operations in $F$ and generated by sets open in the original topology. Similarly, a non-Archimedean metric on $X$ can be extended to a new non-Archimedean metric on $U$. In this new metric, every ball is a subset of $U$ generated by a ball in the original metric and closed with respect to the operations in $F$.

On the base of the topology and metric constructed on $U$, we then construct a topology and metric on the family $L$ of all finite-rank subsets $V \subset U$ closed with respect to the operations in $F$. It is proved that, with respect to the topology thus constructed, the family $L_0$ of planar subsets paying the role of planes in our geometry forms an open dense set.

## 1. $A$-SYSTEMS

### 1.1. *Definition of A-System*

By an *$A$-system* we mean a pair $A = (U, F)$ consisting of a set $U$ of elements (the *support* of the $A$-system) and a set $F$ of operations $f\colon U^{\times n} \to U$ (the *fundamental* set). For any operation $f$, the parameter $n = n(f)$ can be an arbitrary nonnegative integer, which is called the *arity* of the operation $f$, and $f$ itself is called an *$n$-ary operation.* The subset of all $n$-ary operations is denoted by $F_n$. By definition, each 0-ary operation fixes some element of the set $U$.

The image of an arbitrary sequence $(u_1, \dots, u_n) \in U^{\times n}$ under an $n$-ary operation $f$ is denoted by $f(u_1, \dots, u_n)$.

**Example.** 1) A set $U$ with a chosen set of mappings $f\colon U \to U$.

2) A set $U$ with a single binary operation (multiplication) $f\colon U \times U \to U$ (a groupoid) [3].

A homomorphism of one $A$-system into another and an isomorphism of two $A$-systems are defined in the usual way.

The definition of $A$-system coincides with the traditional definition of a universal algebra [1, 2], and, if additional relations are defined on the elements of the support, with the definition of an algebraic system. However, the last term was recently overloaded by other interpretations in various areas of mathematics. Since, in the present paper, we foresee introducing some additional relations, we use a special term "$A$-system" to avoid varying readings. In this paper, we assume that all $A$-systems under consideration contain no 0-ary operations.

### 1.2. *F-Subsets and A-Subsystems*

Let $A = (U, F)$ be an arbitrary $A$-system. Any subset $U' \subset U$ closed with respect to the operations $f \in F$ is said to be *$F$-closed* or, briefly, an *$F$-subset.* By definition, an empty set is assumed to be $F$-closed. The $A$-systems of the form $A' = (U', F)$, where $U' \subset U$ is an arbitrary $F$-subset, are said to be *$A$-subsystems* (or simply *subsystems*) of the original $A$-system $A = (U, F)$.

We say that a subsystem $(U', F)$ is *embedded* in a subsystem $(U'', F)$ if $U' \subset U''$. If all subsets $U_\alpha \subset U$ are $F$-subsets, then their intersection $\cap U_\alpha$ is also an $F$-subset, and the subsystem $(\cap U_\alpha, F)$ is called the *intersection* of the subsystems $(U_\alpha, F)$. Hence, the family of all $F$-subsets $U' \subset U$ and the set of subsystems $A' = (U', F)$ of every $A$-system $A = (U, F)$ are endowed with the structures of lattices with respect to embedding. In these lattices, the products (the compositions) $U_1 \wedge U_2$ and $A_1 \wedge A_2$ of $F$-sets $U_1$ and $U_2$ and subsystems $A_1 = (U_1, F)$ and $A_2 = (U_2, F)$ are the set $U_1 \cap U_2$ and the subsystem $(U_1 \cap U_2, F)$, respectively, and the sums (disjunctions) $U_1 \vee U_2$ and $A_1 \vee A_2$ are the intersection $U'$ of all $F$-subsets containing $U_1$ and $U_2$ and the subsystem $A' = (U', F)$, respectively.

For any $A$-system $A = (U, F)$, denote by $f(U')$, where $U' \subset U$ and $f \in F$, the image of the set $U^{\times n}$, $n = n(f)$, under the mapping $U^{\times n} \to U$.

To any subset $X \subset U$ we assign the sequence of subsets

$$X_1, \dots, X_n, \dots, \tag{3}$$

where $X_1 = X$, and the sets $X_n$ are defined by induction on $n$,

$$X_n = \bigcup_{f \in F} f((X_1 \cup \dots \cup X_{n-1})^{n(f)}).$$

Obviously, their union

$$U' = \bigcup_{n=1}^{\infty} X_n$$

is an $F$-subset, and therefore the pair $A' = (U', F)$ is an $A$-subsystem of $A$. This subsystem is said to be *generated* by the subset $X \subset U$. We then write

$$U' = U(X), \qquad A' = A(X).$$

The set $X$ is said to be a *generating* subset of the $F$-subset $U'$ and of the subsystem $A'$.

A subsystem is said to be *finitely generated* if it admits a finite generating subset.

We say that $X \subset U$ is a *base* subset (in other words, is a *base* or a *base*) of an $A$-system $A = (U, f)$ if $A$ is a generated set $X$ and any proper subset $X' \subset X$ is not a generating set for $A$.

If $X$ is a base subset of an $A$-system $A = (U, F)$, then we write $U[X]$ and $A[X]$ instead of $U(X)$ and $A(X)$, respectively.

Note that every finitely generated $A$-system has a finite base.

## 1.4. *Height and Length of Elements*

Let $A(X) = (U, F)$ be an $A$-system generated by a set $X \subset U$. By the *height* of an element $u \in U$ with respect to $X$ we mean the number $h_X(u)$ that is the least of the positive integers $n$ for which $u \in X_n$, where $\{X_n\}$ is the sequence (3) defined in Subsection 1.3. It follows from the definition that

1) $h_X(u) = 1$ if and only if $u \in X$;

2) if $h_X(u) = n > 1$, then the element $u$ can be represented in the form $u = f(u_1, \dots, u_k)$, where $\max(h_X(u_1), \dots, h_X(u_k)) = n - 1$.

Let us define the *length* $l_X(u)$ of an element $u \in U$ by induction on $h_X(u)$. If $h_X(u) = 1$, then we assume that $l_X(u) = 1$. If $h_X(u) > 1$, then we consider all representations of $u$ in the form

$$u = f(u_1, \dots, u_{n(f)}), \qquad f \in F,$$

where $h_X(u_i) < h_X(u)$, $i = 1, \dots, n(f)$, and set

$$l_X(u) = \min(l_X(u_1) + \dots + l_X(u_{n(f)})),$$

where the minimum is taken over all representations of this kind.

By induction on $h_X(u)$, we obtain

**Proposition 1.1.** *If an $A$-system has no unary operations, then $h_X(u) \leqslant l_X(u)$ for any $u \in U$.*

**Remark.** This is not the case if unary operations exist. For example, consider the infinite set $U = \{u_1, \dots, u_n, \dots\}$ with a single unary operation $f(u_n) = u_{n+1}$. Then we have $h_{\{u_1\}}(u_n) = n$ and $l_{\{u_1\}}(u_n) = 1$ for any $n$.

## 1.5. *Indecomposability Condition for Elements*

An element $u \in U$ of an $A$-system $A = (U, F)$ is said to be *indecomposable* if it cannot be represented in the form $u = f(u_1, \ldots, u_{n(f)})$, where $u_i \neq u$ for at least one index $i$.

By definition, if an element $u$ admits a representation of the form $u = f(u, \ldots, u)$, this is not a decomposition.

Note that there are $A$-systems which have no indecomposable elements. For example, consider a set $U$ of three elements $x_1, x_2, x_3$ with a single binary operation (multiplication):

$$x_i x_i = x_i, \qquad i = 1, 2, 3; \qquad x_i x_j = x_k$$

for any pairwise distinct indices $i, j, k$.

The following assertion results from the definition of indecomposability.

**Proposition 1.2.** *Each subset $U' \subset U$ generating an $A$-system $(U, F)$ contains all indecomposable elements of this $A$-system.*

**Corollary.** *If a subset $X \subset U$ of indecomposable elements generates the $A$-system $(U, F)$, then $X$ is a base, and this base of the $A$-system is unique.*

## 1.6. *$N$-Systems*

An $A$-system $A = (U, F)$ is referred to as an *$N$-system* if it is generated by a subset $X \subset U$ of indecomposable elements. In this case, $X$ is a base, and this base of the $A$-system $A$ is unique; in particular, $A = A[X]$.

The cardinality of the set $X$ is called the *rank* of the $N$-system $A[X]$ and is denoted by

$$r(A[X]) = r(A), \qquad r(A[X]) = \#X.$$

Denote by $h(u)$ and $l(u)$ the height and the length of an element $u$ of an $N$-system with respect to its base.

**Proposition 1.3.** *Each subsystem $A' = (U', F)$ of an $N$-system $A[X] = (U, F)$ is an $N$-system.*

**Proof.** Let $\{X_n\}$ be the sequence of subsets $X_n \subset U$ introduced in Subsection 1.3. We define a sequence of subsets $Y_n \subset U'$ by induction on $n$. Set $Y_1 = X_1 \cap U'$. Let $Y_1, \ldots, Y_{n-1}$ be already defined. Let us define $Y_n$ as the subset of elements in $X_n \cap U'$ that do not belong to the subset $U(Y_1 \cup \cdots \cup Y_{n-1}) \subset U'$ generated by $Y_1 \cup \cdots \cup Y_{n-1}$. It follows from the definition that the elements of the subset $Y = \bigcup_{n=1}^{\infty} Y_n$ are indecomposable in $A' = (U', F)$ and generate $A'$.

In what follows, we consider only $N$-systems unless otherwise stated explicitly.

## 1.7. *Commutative and Idempotent A-Systems*

An $A$-system $A = (U, F)$ is said to be *commutative* if, for any operation $f \in F$ and any $u_1, \ldots, u_{n(f)}$, the element $u = f(u_1, \ldots, u_{n(f)})$ is preserved under all permutations of the elements $u_1, \ldots, u_{n(f)}$.

An $A$-system $A = (U, F)$ is said to be idempotent if $f(u, \ldots, u) = u$ for any $u \in U$ and any operation $f \in F$.

Note that if $A = (U, F)$ is an idempotent $A$-system, then $(\{u\}, F)$ is a subsystem of $A$ for any $u \in U$.

## 2. TAME $A$-SYSTEMS

### 2.1. *Free $A$-Systems*

An $A$-system $A(X) = (U, F)$ generated by a subset $X \subset U$ is said to be *free* if

1) the relation $f(u_1, \dots, u_{n(f)}) = f'(u'_1, \dots, u'_{n(f')})$ holds if and only if $f = f'$ (and hence $n(f) = n(f') = n$) and $u_i = u'_i$, $i = 1, \dots, n$;

2) no element $u \in X$ can be represented in the form $u = f(u_1, \dots, u_{n(f)})$.

In particular, $A$ is an $N$-system and $X$ is its unique base consisting of indecomposable elements.

The simplest example of a free $A$-system is given by a sequence $\{x_1, \dots, x_n, \dots\}$ with the single unary operation $f(x_n) = x_{n+1}$. The base of this $A$-system is $X = \{x_1\}$.

The following proposition results from the definition and Proposition 1.3.

**Proposition 2.1.** *Each subsystem of a free $A$-system is a free $A$-system.*

Let us note the following obvious fact.

**Proposition 2.2.** *In any free $A$-system $A = (U, F)$, to each element $x \in U$ of height $n$, there corresponds an injective mapping $\tau_x \colon F \to X^{(n+1)}$, where $X^{(n+1)} \subset U$ is a subset of elements of height $n + 1$ defined by the formula*

$$\tau_x f = f(x, \dots, x).$$

**Proposition 2.3.** *Two free $A$-systems $A[X] = (U, F)$ and $A'[X'] = (U', F')$ are isomorphic if and only if $r(A) = r(A')$ (i.e., $\#X = \#X'$) and $\#F_n = \#F'_n$, $n = 1, 2, \dots$, where $F_n \subset F$ and $F'_n \subset F'$ are the subsets of $n$-ary operations.*

**Proof.** In one direction, the assertion is obvious: if $A$-systems $A$ and $A'$ are isomorphic, then the condition of the proposition is satisfied. Conversely, let $\#X = \#X'$ and $\#F_n = \#F'_n$, $n = 1, 2, \dots$; then there exist bijections $\sigma \colon X \to X'$ and $\tau \colon F_n \to F'_n$, $n = 1, 2, \dots$ Let us extend the bijection $\sigma$ to a bijection $\sigma \colon U \to U'$ by induction on $h(u)$. Let $\sigma$ be already defined for elements of height less than $n$, and let $h(u) = n$, where $n > 1$. By definition, the element $u$ can be represented in the form

$$u = f(u_1, \dots, u_{k(f)}), \quad f \in F,$$

where $h(u_i) < n$, $i = 1, \dots, k(f)$. Then we set

$$\sigma u = (\tau f)(\sigma u_1, \dots, \sigma u_{k(f)}).$$

It follows from the construction that $\sigma \colon U \to U'$ and $\tau \colon F \to F'$ define an isomorphism of the $A$-systems $A[X]$ and $A'[X']$.

**Simplest examples. 1.** The support of any given free $A$-system with a single unary operation is the disjoint union of some set of sequences $\{x_1, \dots, x_n\}$, and this set can be arbitrary. The unary operation is $F \colon f(x_n) = f(x_{n+1})$.

**2.** The support of a free groupoid (an $A$-system with one binary operation) generated by one element $x$ consists of all words of the form $x$, $xx$, $x(xx)$, $(xx)x$, $(xx)(xx)$, etc.

Note that this groupoid contains subgroupoids with finite base having arbitrarily many elements and even with a countable base. Namely, the sequence $\{x_1, \dots, x_n, \dots\}$, where $x_1 = xx$ and $x_{n+1} = x_n x$, forms a base in the subgroupoid generated by this sequence.

### 2.2. *Embedding in a Free Groupoid*

Consider a special case of a free $A$-system given by a free groupoid, i.e., a set with one binary operation which we call *multiplication.*

According to the general definition, a groupoid $G$ generated by a subset $X \subset G$ is said to be *free* if

1) the relation $x_1 y_1 = x_2 y_2$ holds if and only if $x_1 = x_2$ and $y_1 = y_2$;

2) no element $x \in X$ is representable in the form of a product $x = x_1 y_1$.

Let $A[X] = (U, F)$ be a free $A$-system with base $X \subset U$. Let us construct an embedding

$$U \to G[X \cup F]$$

by induction on $h(u)$, where $G[X \cup F]$ is a free groupoid with base $X \cup F$. If $h(u) = 1$, i.e., $u \in X$, then we set $\sigma(u) = u$. Let $\sigma(u)$ be already defined for all elements of height less than $n$.

Let us first define a mapping $\tau_k\colon (U_n)^{\times k} \to G[X \cup F]$, where $U_n \subset U$ is the subset of elements of height less than $n$, by induction on $k = 1, 2, \dots$ Namely, we set

$$\tau_1(u) = \sigma(u), \qquad \tau_k(u_1, \dots, u_k) = \tau_{k-1}(u_1, \dots, u_{k-1})\,\sigma(u_k).$$

Note that the mappings $\tau_k\colon (U_n)^{\times k} \to G[X \cup F]$ agree with the mappings $\tau_k\colon (U_m)^{\times k} \to G[X \cup F]$ for $m < n$.

Let $h(u) = n$. The element $u$ has a representation in the form $u = f(u_1, \dots, u_{k(f)})$, where $f \in F$, $h(u_i) < h(u)$, $i = 1, \dots, k(f)$, and this representation is unique. Set

$$\sigma(u) = f \cdot \tau_{k(f)}(u_1, \dots, u_{k(f)}).$$

Obviously, the mapping $\sigma\colon U \to G[X \cup F]$ is injective, and the original free $A$-system $A[X]$ can uniquely be reconstructed, up to isomorphism, from the set $\sigma(U) \subset G[X \cup F]$.

### 2.3. *Free Commutative and Free Idempotent A-Systems*

A commutative $A$-system $A(X) = (U, F)$ generated by a subset $X \subset U$ is said to be *free commutative* if

1) the relation $f(u_1, \dots, u_{n(f)}) = f'(u'_1, \dots, u'_{n(f')})$ holds if and only if $f = f'$ (and hence $n(f) = n(f') = n$) and the sequences $(u_1, \dots, u_{n(f)})$ and $(u'_1, \dots, u'_{n(f')})$ coincide up to permutation;

2) no element $u \in X$ can be represented in the form $u = f(u_1, \dots, u_{n(f)})$.

An idempotent $A$-system $A(X) = (U, F)$ generated by a subset $X \subset U$ is said to be *free idempotent* if

1) the relation $f(u_1, \dots, u_{n(f)}) = f'(u'_1, \dots, u'_{n(f')})$ in which some elements of the form $u_i$ are distinct from one another and some elements of the form $u'_i$ are distinct from one another holds if and only if $f = f'$ (and hence $n(f) = n(f') = n$) and $u_i = u'_i$, $i = 1, \dots, n(f)$;

2) no element $u \in X$ can be represented in the form $u = f(u_1, \dots, u_{n(f)})$, where $u_i \neq u$ for at least one index $i$.

In particular, free commutative and free idempotent $A$-systems are $N$-systems, and their generating subsets $X$ are uniquely determined bases formed by indecomposable elements.

Note that the support of a free idempotent system with a single base element consists of this element only.

The analogs of the assertions obtained for free $A$-systems in Subsection 2.1 hold for any free commutative and free idempotent $A$-system.

**Simplest examples. 1.** The support of the free commutative groupoid generated by a single element $x$ consists of the words $x$, $x^2$, $xx^2$, $x(xx^2)$, $(xx^2)x^2$, etc.

**2.** The support of a free idempotent groupoid generated by two elements $x_1$ and $x_2$ consists of the words $x_i$, $x_ix_j$, $x_k(x_ix_j)$, $(x_ix_j)x_k$, where $i \neq j$, $(x_ix_j)(x'_ix'_j)$, where $(i, j) \neq (i', j')$, etc.

Each of these groupoids contains subgroupoids with finite base having arbitrarily many elements and even subgroupoids with countable base.

### 2.4. *Tame A-Systems*

An $A$-system $A = (U, F)$ generated by a subset $X \subset U$ is said to be *tame* if

1) $A$ is an $N$-system, i.e., the set $U$ is generated by a subset $X \subset U$ of indecomposable elements;

2) the following uniqueness condition holds for the decompositions:

for any decomposable element $u \in U$, there is a unique representation (up to the order of the elements $u_i$) of the form

$$u = f(u_1, \dots, u_{n(f)}),$$

where $u_i \neq u$ for at least one index $i$.

In a tame $A$-system any two representations $u = f(u_1, \dots, u_{n(f)})$ and $u = f(u'_1, \dots, u'_{n(f)})$, where $(u_1, \dots, u_{n(f)})$ and $(u'_1, \dots, u'_{n(f)})$ differ on the order only, are regarded as the same decomposition.

It follows from the uniqueness condition for the decompositions that the following property holds.

**Proposition 2.4.** *In a tame $A$-system, it follows from a decomposition*

$$u = f(u_1, \dots, u_n), \quad \textit{where} \quad h(u_i) < h(u), \quad i = 1, \dots, n,$$

*that*

$$h(u) = \max(h(u_1), \dots, h(u_n)) + 1; \qquad l(u) = l(u_1) + \dots + l(u_n).$$

If an idempotent $A$-system is free, free commutative, or free idempotent, then it is a tame $A$-system.

In general, in an arbitrary tame $A$-system, the permutation condition and the idempotent condition hold only for some subsets of operations and elements. Namely, let $A[X] = (U, F)$ be an arbitrary tame $A$-system. For any $f \in F$ and $u_1, \dots, u_{n(f)} \in U$, write

$$U(f) = \{u \in U \mid u = f(u, \dots, u)\},$$

and denote by $M(f;\, u_1, \dots, u_{n(f)})$ the subset of permutations (arrangements) $u'_1, \dots, u'_{n(f)}$ of the elements of $u_1, \dots, u_{n(f)}$ for which

$$f(u_1, \dots, u_{n(f)}) = f(u'_1, \dots, u'_{n(f)}).$$

The following assertion immediately results from the definition of tame $A$-system.

**Proposition 2.5.** *Each tame $A$-system $A[X] = (U, F)$ is uniquely defined up to isomorphism by the sets $U(f)$ and $M(f;\, u_1, \dots, u_{n(f)})$.*

Note that, for any tame $A$-system $A = (U, F)$, one can define a natural surjection $U' \to F$, where $U' \subset U$ is the subset of all decomposable elements. Namely, to any element $u \in U'$, there corresponds an element $f \in F$ defining the decomposition of $u$, $u = f(u_1, \dots, u_n)$.

**Proposition 2.6.** *Every subsystem of a tame $A$-system is a tame $A$-system.*

As in the case of a free $A$-system, the assertion immediately follows from Proposition 1.3 and from the definition of tame $A$-system.

### 2.5. *Subordination Relation*

Let us introduce a subordination relation on elements of the support $U$ of an arbitrary tame $A$-system $A = (U, F)$.

**Definition.** We say that an element $u' \in U$ is *immediately subordinated* to a decomposable element $u$, $u \neq u'$, if one can find a sequence $\{u_1, \dots, u_n\} \subset U$ containing $u'$ and an $n$-ary operation $f \in F$ such that

$$u = f(u_1, \dots, u_n).$$

It follows from the uniqueness of the decomposition that, for any decomposable element $u \in U$, the set of elements immediately subordinated to $u$ is finite and nonempty.

**Definition.** We say that an element $u' \in U$ is *subordinated* to an element $u \in U$ if either $u' = u$ or $u$ is decomposable and there exists a finite sequence $u' = u_1, u_2, \ldots, u_n = u$ of decomposable elements in which each element except for the first one is immediately subordinated to the preceding element.

According to this definition, if $(U', F)$ is a subsystem of $(U, F)$, $u', u'' \in U'$, and $u''$ is subordinated to $u'$ in the subsystem $(U', F)$, then $u''$ is subordinated to $u'$ in the $A$-system $(U, F)$ as well.

The following assertion also results from the definitions.

**Proposition 2.7.** *If an element $u'$ is subordinated to an element $u$, $u \neq u'$, then*

$$h(u') < h(u), \qquad l(u') \leqslant l(u).$$

*If the sequence of decomposable elements from $u'$ to $u$ involves at least one operation $f$ satisfying $n(f) > 1$, then $l(u') < l(u)$, and otherwise $l(u') = l(u)$.*

**Corollary.** *For any $u \in U$, the set of elements subordinated to $u$ is finite.*

### 2.6. *Partial Order Relations on the Supports of Tame A-Systems*

Since the relations $h(u') \leqslant h(u)$ and $l(u') \leqslant l(u)$ and the subordination relation are transitive, it follows that each of them induces a partial order relation on the support $U$ of a given tame $A$-system. Namely, we write

$$\begin{aligned} &u' \underset{h}{\leqslant} u \qquad \text{if} \quad h(u') \leqslant h(u), \\ &u' \underset{l}{\leqslant} u \qquad \text{if} \quad l(u') \leqslant l(u), \\ &u' \leqslant u \qquad \text{if} \quad u' \quad \text{is subordinated to} \quad u. \end{aligned}$$

The first of these partial order relations is stronger than the other two, i.e., it follows from $u' \leqslant u$ that $u' \leqslant_l u$ and $u' \leqslant_h u$; the converse assertion fails. The partial order relations $\leqslant_l$ and $\leqslant_h$ are not compatible. For instance, in the free groupoid $G[a, b, c, d, e, f]$, there are pairs of elements for which either these relations hold or their opposite relations hold; for example, if $x = (((ab)c)d)e$, $y = ab$, and $z = ((ab)(cd))(ef)$, then $h(x) = 4$, $l(x) = 5$, $h(y) = 1$, $l(y) = 2$, $h(z) = 3$, and $l(z) = 6$.

Note that the relations $\leqslant_h$ and $\leqslant_l$ are invariant with respect to the operations $f \in F$, namely, if $f$ is an $n$-ary operation, then it follows from the assumptions $u'_i \leqslant_h u_i$, $i = 1, \ldots, n$, that $f(u'_1, \ldots, u'_n) \leqslant f(u_1, \ldots, u_n)$, and the same holds for the relation $\leqslant_l$.

The relation $\leqslant$ is in general not invariant with respect to the operations $f \in F$. For instance, in the free groupoid generated by elements $x, y, z$, we have $x \leqslant xy$, but $xz$ is not subordinated to $(xy)z$.

### 2.7. *Graphs Associated with Tame A-Systems*

The subordination relation induces the structure of a directed graph in the support $U$ of a given tame $A$-system. The vertices of this graph are the elements $u \in U$, and the directed edges join each element $u$ with all elements immediately subordinated by $u$, i.e., with the elements $u_i$ entering the decomposition $u = f(u_1, \ldots, u_n)$. Moreover, if $u_i$ enters the decomposition $n$ times, then there are $n$ edges coming from $u$ to $u_i$. Note that, for each vertex of the graph, there are only finitely many edges issuing from this vertex and infinitely many incoming edges.

**Proposition 2.8.** *Every free $A$-system $A = (U, F)$ is uniquely defined by the associated graph $G$ (up to isomorphism).*

**Proof.** By Proposition 2.3, it suffices to prove that the cardinalities of the set $X$ and of the set $F_n \subset F$ of the $n$-ary operations are uniquely determined by the graph $G$.

The assertion related to $X$ is obvious because the set $X$ coincides with the set of graph vertices from which no edges issue.

Let us now choose an arbitrary element $x \in X$ and consider, for any $n$, the mapping $F_n \to U$ taking every operation $f \in F_n$ to the element $f(x, \dots, x) \in U$. This mapping is bijective. On the other hand, the elements $f(x, \dots, x)$ are those and only those graph vertices from which exactly $n$ edges issue, and all these edges end at the point $x$. Thus, for any $n$, the cardinality of the set $F_n$ is also uniquely defined by the graph $G$.

### 2.8. *Subsets $X_u$*

Let $X$ be the base of a tame $A$-system $A = (U, F)$. To any element $u \in U$ we assign the subset

$$X_u = \{x \in X \mid x \leqslant u\}.$$

In particular, if $u \in X$, then $X_u = \{u\}$.

The definition implies the following assertion.

**Proposition 2.9.** *An element $u$ belongs to the support $U' \subset U$ of a subsystem $A[X']$ with base $X' \subset X$ if and only if $X_u \subset X'$.*

Proposition 2.7 implies the following assertion.

**Proposition 2.10.** *If $u = f(u_1, \dots, u_{n(f)})$, then*

$$X_u = X_{u_1} \cup \dots \cup X_{u_{n(f)}}. \tag{4}$$

**Proposition 2.11.** *Let $A = (U, F)$ be a tame $A$-system, let $U' \subset U$ be an $F$-subset, and let $X \subset U'$ be the base of $U'$. Then*

$$h(u) > h(x)$$

*for any $u \in U'$ and $x \in X_u$, $x \neq u$, where $l$ stands for the length of elements in $U$.*

**Corollary.** *Let $U'$ and $U''$ be $F$-subsets with bases $X' \subset U'$ and $X'' \subset U''$. Then, if $x' \in X' \cap U''$ and $x'' \in X'' \cap U'$, $x' \neq x''$, then the relations $x'' \in X''_{x'}$ and $x' \in X'_{x''}$ cannot hold simultaneously.*

Indeed, otherwise the relations $h(x') > h(x'')$ and $h(x'') > h(x')$ would hold simultaneously.

**Proposition 2.12.** *In the notation of the previous proposition, for any $u \in U'$, we have*

$$l(u) \geqslant \sum_{x \in X_u} l(x). \tag{5}$$

*Moreover, if $u = f(u_1, \dots, u_{n(f)})$, where $u_i \in U'$, and if at least two sets $X_{u_i}$ have nonempty intersection, then inequality (5) is strict.*

**Proof.** (We proceed by induction on $h_X(u)$.) The assertion is obvious if $h_X(u)) = 1$. Assume that $h_X(u)) = n > 1$; then $u = f(u_1, \dots, u_{n(f)})$, where $u_i \in U'$, and $h_X(u_i) < h_X(u)$ for any $i$. By the induction assumption,

$$l(u_i) \geqslant \sum_{x \in X_{u_i}} l(x).$$

Therefore, the assertion for $u$ immediately follows from (4) and from the relation

$$l(u) = l(u_1) + \dots + l(u_{n(f)}).$$

**Corollary 2.1.** $l(u) \geqslant \#X_u$.

**Corollary 2.2.** *If $\#X_u > 1$, then $l(u) > l(x)$ for any $x \in X_u$.*

### 2.9. *Completeness Condition*

Let $U[Y]$ and $U[Z]$ be $F$-subsets of a tame $A$-system, and let $U[Y] \subset U[Z]$. We say that $U[Y]$ is *complete* in $U[Z]$ if $U[Y] \not\subset U[Z']$ for any proper subset $Z' \subset Z$.

The following assertion results from the definition of the sets $X_u$.

**Proposition 2.13.** *$U[Y]$ is complete in $U[Z]$ if and only if*

$$Z = \bigcup_{y \in Y} Z_y.$$

**Proposition 2.14.** *If $U[Y] \subset U[Z]$, then $U[Y]$ is a complete subset of $U[Z'] \subset U[Z]$, where*

$$Z' = \bigcup_{y \in Y} Z_y.$$

*Moreover, if these subsets $U[Y]$ and $U[Z]$ are of finite rank, if $r(U[Y]) \geqslant r(U[Z])$, and if $U[Y]$ is strictly contained in $U[Z]$, then the set $U[Y]$ is also strictly contained in $U[Z']$.*

**Proof.** The first assertion immediately follows from the definition of completeness. The other assertion is obvious if $Z' = Z$. If $Z'$ is strictly contained in $Z$, then $r(U[Z']) < r(U[Z])$. Thus, $r(U[Z']) < r(U[Y])$, and therefore $U[Z'] \neq U[Y]$.

**Proposition 2.15.** *If $U[Y]$ and $U[Z]$ are $F$-subsets of finite rank and if $U[Y]$ is complete in $U[Z]$, then*

$$\sum_{y \in Y} l(y) \geqslant \sum_{z \in Z} l(z).$$

*Moreover, if at least two subsets $Z_y$, $y \in Y$, have nonempty intersection, then the inequality is strict.*

**Proof.** By Proposition 2.10,

$$l(y) \geqslant \sum_{z \in Z_y} l(z)$$

for any $y \in Y$. Hence,

$$\sum_{y \in Y} l(y) \geqslant \sum_{y \in Y} \sum_{z \in Z_y} l(z).$$

Since

$$Z = \bigcup_{y \in Y} Z_y$$

by the completeness condition, this immediately proves the proposition.

### 2.10. *Ascending Chain Condition*

Consider an arbitrary ascending sequence (not necessarily strictly ascending) of finite-rank $F$-subsets of a tame $A$-system,

$$U[Y_1] \subset \cdots \subset U[Y_n] \subset \cdots \tag{6}$$

**Proposition 2.16.** *If $U[Y_n]$ is complete in $U[Y_{n+1}]$ and $r(U[Y_n]) \geqslant r(U[Y_{n+1}])$ for any $n$, then the sequence* (6) *stabilizes at a finite step.*

**Proof.** Assume the contrary. Let there exist a strictly ascending sequence (6) satisfying the conditions of the proposition. By Proposition 2.15,

$$\sum_{y_n \in Y_n} l(y_n) \geqslant \sum_{y_{n+1} \in Y_{n+1}} l(y_{n+1})$$

for any $n$. Since, for this sequence, the ranks $r(U[Y_n])$ and the sums $\sum_{y_n \in Y_n} l(y_n)$ are stabilized at a finite step, we can assume that these ranks and sums are the same for all terms of the sequence.

Consider an arbitrary pair $Y' = Y_n$ and $Y'' = Y_{n+1}$. If at least two subsets $Y''_y$, $y \in Y'$, have a nonempty intersection, then $\sum_{y \in Y'} l(y) > \sum_{y \in Y''} l(y)$ (by Proposition 2.15), which contradicts the assumption. Thus, the subsets $Y''_y$, $y \in Y'$, are pairwise disjoint. In this case, since $\sum_{y \in Y'} l(y) = \sum_{y \in Y''} l(y)$ and $Y'' = \sum_{y \in Y'} Y''_y$, we see that all sets $Y''_y$ are singletons, $Y''_y = \{z_y\}$; moreover, $l(y) \geqslant l(z_y)$, and the mapping $y \to z_y$ is a bijection $Y' \to Y''$. If $l(y) = l(z_y)$ for any $y \in Y'$, then $y = z_y$, and therefore $U[Y'] = U[Y'']$, which contradicts the assumption. If $l(y) > l(z_y)$ for at least one $y \in Y'$, then $\sum_{y \in Y'} l(y) > \sum_{y \in Y''} l(y)$, which also contradicts the assumption.

**Theorem 2.1.** *For a tame A-system, every sequence* (6) *of F-subsets satisfying the inequality* $r(U[Y_n]) \geqslant r(U[Y_{n+1}])$ *for any n stabilizes at a finite step.*

**Proof.** Suppose the contrary. Let there exist a strictly ascending sequence (6) satisfying the conditions of the theorem. It follows from Proposition 2.13 that, in this case, there exists a strictly ascending sequence (6) such that $U[Y_n]$ is complete in $U; Y_{n+1}$ and $r(U[Y_n]) \geqslant r(U[Y_{n+1}])$ for any $n$. This contradicts Proposition 2.16.

## 2.11. *Decomposition Schemes*

Let $A = (U, F)$ be a free $A$-system. To each element $u \in U$ we assign a directed graph $S(u)$ of tree type; we call this graph the *decomposition scheme of the element* $u$ (or, briefly, the *scheme of* $u$). The vertex of this graph without incoming edges is called a *root*, and the vertices from which no edges issue are referred to as *leaves*. Let us define the graph $S(u)$ by induction on the height $h(u)$. If $h(u) = 1$, i.e., if $u \in X$, where $X$ is the base, then, by definition, $S(u)$ consists of one point which is the root and a leaf simultaneously. If $h(u) = n > 1$, then let us represent $u$ in the form $u = f(u_1, \ldots, u_n)$, $n = n(f)$, where $h(u_i) < h(u)$, $i = 1, \ldots, n$. In this case, by definition, the scheme $S(u)$ is obtained from the schemes $S(u_1), \ldots, S(u_n)$ by adding one vertex (the root of the scheme $S(u)$) and $n$ edges coming from this root to the roots of the schemes $S(u_1), \ldots, S(u_n)$. In this case, the root and the edges are equipped with labels, namely, the root has the label $f$ (the symbol of the corresponding operation) and the edges are equipped with the digits $1, \ldots, n$ (the indices of the elements $u_i$ in the sequence $(u_1, \ldots, u_n)$).

This definition of decomposition scheme still makes sense for free idempotent $A$-systems $A = (U, F)$ because, for these systems, the decomposable elements $u \in U$ can also be represented uniquely in the form

$$u = f(u_1, \ldots, u_n), \quad \text{where} \quad h(u_i) < h(u), \quad i = 1, \ldots, n.$$

The definition of decomposition schemes can also be extended to arbitrary tame $A$-systems. Here it is assumed that the labels at the edges contain additional information concerning the list of admissible permutations of elements in the expression $u = f(u_1, \ldots, u_n)$. If the $A$-system in question is commutative, then we put no labels at the edges.

In terms of schemes, the height $h(u)$ of an element $u$ is equal to the maximal number of tiers of the scheme $S(u)$ and the length to the number of leaves of the scheme. Thus, if $S(u) = S(v)$, then $h(u) = h(v)$ and $l(u) = l(v)$.

**Example.** Let $A = (U, F)$ be the free $A$-system with a single binary operation (a free groupoid). In Figures 1, 2, and 3, we show the schemes of the elements $(x_1x_2)(x_3x_4)$, $(x_1(x_2x_3))x_4$, and $x_1((x_2x_3)x_4)$, respectively, where $x_i$ are arbitrary (not necessarily pairwise disjoint) elements of the base $X \subset U$.

**Definition.** By *S-subsets* of a tame $A$-system $A = (U, F)$ we mean subsets $V \subset U$ formed by elements with the same decomposition scheme.

According to this definition, the set $X$ of all indecomposable elements (i.e., elements of unit height) is an $S$-subset, and all other $S$-subsets are defined by induction on the height of their elements. Namely, let the $S$-subsets consisting of elements of height less than $k$ be already defined. Then any $S$-subset $V$ formed by elements of height $k$ is given by an operation $f \in F$ and a

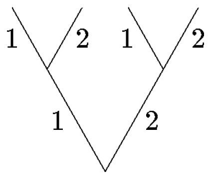


Fig. 1

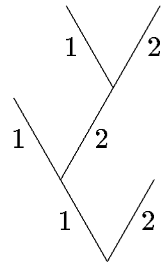


Fig. 2

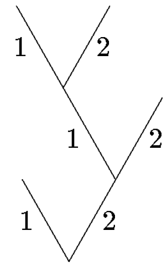


Fig. 3

sequence of $S$-subsets $V_1, \ldots, V_n$, where $n$ is the arity of $f$, such that the heights of the elements of these subsets is less than $k$, and at least one of the subsets consists of elements of height $k-1$. The $S$-subset $V$ is formed by the elements of height $k$ that are representable in the form $v = f(v_1, \ldots, v_n)$, where $v_i \in V_I$, $i = 1, \ldots, n$. In particular, every $S$-subset of elements of height two is defined by an operation $f \in F$ and consists of all elements of height two representable in the form $u = f(x_1, \ldots, x_n)$, where $x_i \in X$, $i = 1, \ldots, n$, and $n$ is the arity of the operation $f$.

**Example.** In any free groupoid the subset of all elements of height two is an $S$-subset, as well as the subset of elements of height one. The family of elements of height three is decomposed into three $S$-subsets. Their representatives are elements of the form $x_1(x_2x_3)$, $(x_1x_2)x_3$, and $(x_1x_2)(x_3x_4)$.

Denote by $H_n$ the number of $S$-subsets of a free groupoid formed by elements of height $n$. It follows from what was said above that $H_1 = H_2 = 1$ and $H_3 = 3$. Let us show that the following recurrence formula holds for any $n > 2$.

**Assertion.**

$$H_n = \left(\frac{H_{n-1}}{H_{n-2}} + H_{n-1} + H_{n-2}\right) H_{n-1}.$$

Indeed, there are three types of $S$-subsets with elements of height $n$, namely, the $S$-subsets with elements of the form $xy$, where 1) $h(x) = h(y) = n-1$, 2) $h(x) = n-1$ and $h(y) < n-1$, and 3) $h(x) < n-1$ and $h(y) = n-1$, respectively. The number of subsets of the first type is $(H_{n-1})^2$, and that of the second (and of the third) type is $(H_1 + \cdots + H_{n-2})H_{n-1}$. Thus,

$$H_n = (H_{n-1})^2 + 2\,(H_1 + \cdots + H_{n-2})\,H_{n-1}.$$

Hence, $H_n/H_{n-1} = 2\,(H_1 + \cdots + H_{n-2}) + H_{n-1}$. From this relation, let us subtract the relation $H_{n-1}/H_{n-2} = 2\,(H_1 + \cdots + H_{n-3}) + H_{n-2}$; we then obtain the desired recurrence relation. In particular, $H_4 = 21$, $H_5 = 651$, etc.

The same recurrence formula defines the number $H_n$ of $S$-subsets with elements of height $n$ in any free idempotent groupoid. We similarly obtain the following assertion.

**Assertion.** *In any free commutative groupoid, the number $H_n$ of $S$-subsets with elements of height $n$ is given for $n > 2$ by the following recurrence formula*:

$$H_n = \left(\frac{H_{n-1}}{H_{n-2}} + \frac{H_{n-1} + H_{n-2}}{2}\right) H_{n-1}.$$

*In particular, $H_3 = 2$, $H_4 = 7$, $H_5 = 112$, etc.*

### 2.12. *A-Systems of S-Subsets*

To any tame $A$-system $A = (U, F)$ we assign another $A$-system $A_S = (\Sigma, F)$ whose support is the family $\Sigma$ of all $S$-subsets of $U$ and the fundamental set coincides with the fundamental set of the original $A$-system.

The action of the operations $f \in F$ on the set $\Sigma$ is introduced in the natural way. Namely, let $f \in F$ be an arbitrary $n$-ary operation, $V_1, \dots, V_n$ are arbitrary $S$-subsets of $U$, and $v_i \in V_i$, $i = 1, \dots, n$, are some representatives of the sets $V_i$. It is assumed that, if some sets $V_i$ coincide, then their representatives coincide as well. By definition, $V = f(V_1, \dots, V_n)$ is the $S$-subset containing the element $v = f(v_1, \dots, v_n)$. The set $V$ is well defined because it does not depend on the choice of the representatives $v_i \in V_i$.

It follows from the definition that

(1) the $A$-system $A_S$ of the $S$-subsets of a tame $A$-system $A$ is also tame;
(2) if $A$ is a free, free commutative, or a free idempotent $A$-system, then the $A$-system $A_S$ of the $S$-subsets of $A$ is also free, free commutative, or free idempotent $A$-system, respectively;
(3) if $A$ is a free or a free commutative $A$-system, then the base of the $A$-system $A_S$ consists of a single element, namely, of the $S$-subset $X$ of elements of unit height;
(4) if $A$ is a free idempotent $A$-system, then the $S$-subset $V$ belongs to the base of the $A$-system $A_S$ (i.e., is an indecomposable element of this $A$-system) if and only if either $V = X$ or the elements $v \in V$ are of the form $v = f(v_1, \dots, v_n)$, where $h(v_i) < h(v)$, and all $v_i$ belong to the same $S$-subset.

**Example.** For any free (free commutative) groupoid, the $A$-system of its $S$-subsets is a free (free commutative) groupoid with a single generating element.

For any free idempotent nontrivial (i.e., not one-element) groupoid, the $A$-system of its $S$-subsets is a free idempotent groupoid with countable base. Namely, an $S$-subset belongs to this base if and only if either it is a base set or the elements of this set can be represented in the form $v = v_1 v_2$, where $h(v_1) = h(v_2) = h(v) - 1$ and $v_1$, $v_2$ belong to the same $S$-subset. Hence, the number of base $S$-subsets consisting of the elements of height $n$ is equal to the number of the $S$-subsets consisting of the elements of height $n - 1$.

The following assertion results from Proposition 2.3.

**Proposition 2.17.** *For any free A-systems $A' = (U', F')$ and $A'' = (U'', F'')$, the A-systems of their S-subsets are isomorphic if and only if $\#F'_n = \#F''_n$, $n = 1, 2, \dots$, where $F'_n \subset F'$ and $F''_n \subset F''$ are the corresponding subsets of the n-ary operations.*

### 2.13. *Secondary A-Systems on Free A-Systems*

Let $A[X] = (U, F)$ be a free $A$-system, and let its base $X$ be equipped with the structure of an $A$-system, i.e., let an arbitrary set $\Phi$ of operations on $X$ be defined. Starting from the $A$-system $(X, \Phi)$, we shall now define an $A$-system $(V, \Phi)$ on each $S$-subset $V \subset U$.

By definition, an $A$-system is given on the $S$-subset of the elements of unit height, i.e., on the subset $X$. Assume that this system is already defined on the $S$-subsets with elements of height less than $k$, and let $V$ be an $S$-subset with elements of height $k$. According to Subsection 2.11, the set $V$ is defined by an operation $f \in F$ and a sequence $V_1, \dots, V_n$ of $S$-subsets, where $n$ is the arity of $f$, whose elements are of height less than $k$, and the set $V$ consists of the elements of the form $v = f(v_1, \dots, v_n)$, where $v_i \in V_I$, $i = 1, \dots, n$.

Let $\varphi \in \Phi$ be an arbitrary operation of arity $m$, and let $u_1, \dots, u_m$ be arbitrary elements of $V$. To define the element $\varphi(u_1, \dots, u_m) \in V$, we represent the elements $u_i$ in the form

$$u_i = f(u_{i1}, \dots, u_{in}), \quad \text{where} \quad u_{ij} \in V_j.$$

By the induction assumption, the corresponding element $v_j = \varphi(u_{1j}, \dots, u_{mj}) \in V_j$ is already defined for any $j$, $j = 1, \dots, n$. Set

$$\varphi(u_1, \dots, u_m) = f(v_1, \dots, v_n).$$

Obviously, $\varphi(u_1, \dots, u_m) \in V$.

**Example.** If $X$ is endowed with the structure of a group $G$, then this structure induces a group structure on any $S$-subset $V \subset U$. The group thus obtained is isomorphic to the direct product of $l$ copies of the group $G$, where $l$ is the length of the elements in $V$.

## 3. LATTICES RELATED TO TAME $A$-SYSTEMS

### 3.1. *Intersections and Unions of $F$-Subsets*

Let $A[X] = (U, F)$ be an arbitrary tame $A$-system with base $X \subset U$ (the subset of indecomposable elements). Consider the family of $F$-subsets $U' \subset U$, i.e., the subsets closed with respect to the operations $f \in F$. Recall that we denote by $U[Y]$ the $F$-subset of $U$ with base $Y$. By Proposition 2.6, $Y$ coincides with the set of indecomposable elements in $U[Y]$. Denote by $r(U[Y])$ the rank of this subset, i.e., $r(U[Y]) = \#Y$.

**Proposition 3.1.** *For any $Y \subset X$ and $Z \subset X$,*

$$U[Y] \cap U[Z] = U[Y \cap Z].$$

**Proof.** Obviously, $U[Y \cap Z] \subset U[Y] \cap U[Z]$. Conversely, if $u \in U[Y] \cap U[Z]$, then $X_u \in Y \cap Z$, and therefore $u \in U[Y \cap Z]$.

**Theorem 3.1.** *The intersection of arbitrary $F$-subsets $U[Y]$ and $U[Z]$ is generated by the subset $Y' \cup Z'$, where $Y' = Y \cap U[Z]$ and $Z' = Z \cap U[Y]$. Thus,*

$$U[Y] \cap U[Z] = U[W],$$

*where $W$ is a subset of the set $Y' \cup Z'$.*

*In particular, if $Y' = Z' = \varnothing$, then $U[Y] \cap U[Z] = \varnothing$.*

**Proof.** Write $B = U(Y' \cup Z')$. It is clear that $B \subset U[Y] \cap U[Z]$. Assume that $U[Y] \cap U[Z] \not\subset B$. Choose an element $u \in (U[Y] \cap U[Z]) \setminus B$ of the minimal possible height $h_Y(u)$. Since $h_Y(u) > 1$ and $h_Z(u) > 1$, the element $u$ can be represented in the form

$$u = f_1(a_1, \dots, a_m) = f_2(b_1, \dots, b_n),$$

where $a_i \in U[Y]$, $h_Y(a_i) < h_Y(u)$ for any $i$, $i = 1, \dots, m$, and $b_j \in U[Z]$, $h_Z(b_j) < h_Z(u)$ for any $j$, $j = 1, \dots, n$. It follows from the uniqueness condition for the decompositions that $f_1 = f_2$, $m = n$, and the sequences $(a_1, \dots, a_m)$ and $(b_1, \dots, b_n)$ coincide up to order. Thus, all elements $a_i$ and $b_j$ belong to $U[Y] \cap U[Z]$. Since $u \notin B$, it follows that $a_i \notin B$ for at least one $i$, for instance, for $a_1 \notin B$. Then $a_1 \in (U[Y] \cap U[Z]) \setminus B$. Since $h_Y(a_1) < h_Y(u)$, this contradicts the assumption that $h_Y(u)$ is minimal.

By the *union* $U_1 \vee U_2$ of $F$-subsets $U_1 = U[Y]$ and $U_2 = U[Z]$ we mean the intersection of the $F$-subsets containing the sets $U_1$ and $U_2$ simultaneously. Obviously, the union $U_1 \vee U_2$ is generated by the subset $Y \cup Z$, and therefore

$$U[Y] \vee U[Z] = U[W], \quad \text{where} \quad W \quad \text{is a subset of} \quad Y \cup Z.$$

**Theorem 3.2.** *Let $U_1 = U[Y]$ and $U_2 = U[Z]$ be $F$-subsets of finite rank of a tame $A$-system, and let $U_1 \vee U_2 = U[W]$, where $W \subset Y \cup Z$. Then*

$$W \subset (Y \setminus Y') \cup (Z \setminus Z') \cup (Y' \cap Z')$$

*for any subsets $Y' \subset Y \cap U[Z]$ and $Z' \subset Z \cap U[Y]$. Hence,*

$$U_1 \vee U_2 = U((Y \setminus Y') \cup (Z \setminus Z') \cup (Y' \cap Z')),$$

*i.e., $U_1 \vee U_2$ is generated by the subset $(Y \setminus Y') \cup (Z \setminus Z') \cup (Y' \cap Z')$.*

**Proof.** Write $Y \setminus Y' = Y^*$ and $Z' \setminus Y' = Q$. Let us first prove that

$$z \in U(Y^* \cup (Z \setminus \{z\})) \quad \text{for any} \quad z \in Q. \tag{7}$$

Let $z \in Q$ and $y \in Y_z \cap Y'$. Since $y \neq z$, it follows from Proposition 2.11 (see the corollary) that $z \notin Z_y$, i.e., $y \in U[Z \setminus \{z\}]$. Hence, $Y_z \cap Y' \subset U[Z \setminus \{z\}]$. Therefore, since $Y_z \subset (Y_z \cap Y') \cup Y^*$, it follows that

$$Y_z \subset U(Y^* \cup (Z \setminus \{z\})) \quad \text{for any} \quad z \in Q.$$

This proves relation (7). It follows from (7) that $U[Z] \subset U(Y^* \cup (Z \setminus \{z\}))$ for any $z \in Q$. Since $Y' \subset U[Z]$, we also have $U[Y] \subset U(Y^* \cup (Z \setminus \{z\}))$. Thus,

$$U(Y \cup Z) = U(Y^* \cup (Z \setminus \{z\})) \quad \text{for any} \quad z \in Q,$$

and hence $W \subset Y^* \cup (Z \setminus \{z\})$ for any $z \in Q$, which implies $W \subset Y^* \cup (Z \setminus Q)$. It remains to note that

$$Y^* \cup (Z \setminus Q) = (Y \setminus Y') \cup (Z \setminus Z') \cup (Y' \cap Z').$$

This completes the proof of the theorem.

**Corollary.**

$$U(Y \cup Z) = U((Y \setminus Y') \cup Z) = U(Y \cup (Z \setminus Z')) = U((Y \setminus Y') \cup (Z \setminus Z') \cup (Y \cap Z)),$$

*where* $Y' = Y \cap U[Z]$ *and* $Z' = Z \cap U[Y]$.

### 3.2. *Lattice of F-Subsets of Finite Rank*

Denote by $L = L(U, F)$ the family of $F$-subsets of finite rank in a tame $A$-system $A = (U, F)$. The set $L$ forms a lattice with respect to the above operations of taking the sum (union) $\vee$ and the product (intersection) $\wedge$. According to 3.1, if $U_1 = U[Y]$ and $U_2 = U[Z]$, then $U_1 \vee U_2 = U(Y \cup Z)$, $U_1 \wedge U_2 = U[W]$, where $W \subset (Y \cap U[Z]) \cup (Z \cap U[Y])$. Thus, the operations of union and intersection over $F$-subsets in $L$ are reduced to related operations over their bases.

**Theorem 3.3.** *The ranks of the F-subsets $U_1$, $U_2$, $U_1 \wedge U_2$, and $U_1 \vee U_2$ are related as follows:*

$$r(U_1 \wedge U_2) + r(U_1 \vee U_2) \leqslant r(U_1) + r(U_2). \tag{8}$$

**Proof.** Let $U_1 = U[Y]$, $U_2 = U[Z]$, and $U_1 \wedge U_2 = U[W]$. Since $W \subset (Y \cap U[Z]) \cup (Z \cap U[Y])$, let us represent $W$ in the form of a disjoint union $W = Y' \cup Z'$, where $Y' \subset Y \cap U[Z]$ and $Z' \subset Z \cap U[Y]$. By Theorem 3.2, then we have

$$U_1 \vee U_2 = U(Y \cup Z) = U((Y \setminus Y') \cup (Z \setminus Z')).$$

Hence,

$$r(U_1 \vee U_2) \leqslant \#[(Y \setminus Y') \cup (Z \setminus Z')].$$

Since

$$\#[(Y \setminus Y') \cup (Z \setminus Z')] \leqslant (\#Y) + (\#Z) - (\#(Y' \cup Z')),$$

it follows that $r(U_1 \vee U_2) \leqslant r(U_1) + r(U_2) - r(U_1 \wedge U_2)$.

**Note.** There are examples of finite-rank $F$-subsets $U_1$ and $U_2$ satisfying the inequalities

$$r(U_1 \wedge U_2) > \max(r(U_1), r(U_2)) \quad \text{and} \quad r(U_1 \vee U_2) < \min(r(U_1), r(U_2)).$$

For instance, let $U$ be the free groupoid generated by some elements $x$ and $y$. Let us define the elements $x_n$ and $y_n$ by induction on $n$,

$$x_1 = x, \qquad y_1 = y, \qquad x_{n+1} = x_n x, \qquad y_{n+1} = y_n y, \qquad n = 1, 2, \dots$$

Let $U_1$ and $U_2$ be the subgroupoids generated by the elements $y, x_1, \dots, x_m$ and $x, y_1, \dots, y_n$, respectively. Then $U_1 \vee U_2 = U$, i.e., the groupoid $U_1 \vee U_2$ is generated by the elements $x$ and $y$, and the groupoid $U_1 \wedge U_2$ is generated by the elements $x_1, \dots, x_m, y_1, \dots, y_n$. Hence, $r(U_1) = m+1$, $r(U_2) = n+1$, $r(U_1 \vee U_2) = 2$, and $r(U_1 \wedge U_2) = m + n$.

Note that, in this example, we have

$$r(U_1 \wedge U_2) + r(U_1 \vee U_2) = r(U_1) + r(U_2).$$

The lattice $L$ is "not geometric" because, in this lattice, the relation $U_1 \subset U$ does not imply the inequality $r(U_1) \leqslant r(U_2)$. Moreover, in general, $F$-subsets of finite rank can contain $F$-subsets of any finite rank and even $F$-subsets of infinite rank. This lattice is not semimodular. For instance, in the free idempotent groupoid $G[x, y]$, the one-element subgroupoids $G_1 = G[x]$ and $G_2 = G[y]$ cover (in the sense of lattice theory) the subgroupoid $G_3 = \varnothing$. However, their union $G[x, y]$ does not cover $G_1$ and $G_2$. For instance, the subgroupoid $G[x, xy]$ contains $G_1$ and is contained in $G[x, y]$.

Let $A = (U, F)$ be an arbitrary tame $A$-system.

**Definition.** We say that an $F$-subset $V \subset U$ has *finite corank* if one can find finitely many elements $u_1, \dots, u_n \in U$ such that the $F$-subset of $U$ generated by the subset $V$ and the elements $u_1, \dots, u_n$ coincides with $U$. The minimal number $r$ of these additional elements is called the *corank* of the $F$-subset $V$ and is denoted by $\operatorname{cor}(V)$.

Note that, if $U$ is of finite rank $n$, then all nonempty $F$-subsets $V \subset U$ have finite corank not exceeding the number $n - 1$.

**Proposition 3.2.** *If $X$ is the base of the support $U$ of an $A$-system, then an $F$-subset $V$ with base $Y$ has finite corank $r$ if and only if $Y$ contains all elements of the base $X$ of $U$ possibly except for finitely many elements, i.e., the set $X \setminus (X \cap Y)$ is finite.*

**Proof.** Obviously, if $Y$ contains all elements of $X$ except for finitely many, then the $F$-set $V$ has finite corank.

Conversely, let $V$ be an $F$-set of finite corank $r$. Then there exist $r$ elements $u_1, \dots, u_r \in U$ such that the $F$-subset generated by $V$ and these elements coincides with $U$. Hence, the base $X$ of the set $U$ is contained in $Y \cup \{u_1, \dots, u_r\}$, i.e., $X \subset Y \cup \{u_1, \dots, u_r\}$. Thus,

$$X = (X \cap Y) \cup (Y \cap \{u_1, \dots, u_r\}). \tag{9}$$

In other words, $Y$ contains all elements of the base $X$ except for finitely many elements of $X$.

**Proposition 3.3.** *For any $F$-subset $V$ of corank $r$, there exists an $r$-tuple of elements $u_1, \dots, u_r$ in $X$ such that the $F$-subset generated by the set $V$ and the elements $u_1, \dots, u_r$ coincides with $U$. This $r$-tuple is uniquely defined and is given by*

$$\{u_1, \dots, u_r\} = X \setminus (X \cap Y), \tag{10}$$

*where $Y$ is the base in $V$.*

The set of elements $u_1, \dots, u_r$ defined by relation (10) is called the *cobase* of the $F$-subset $V$.

**Proof.** Let us apply formula (9). Note that all elements $u_i$ belong to $X$. Indeed, if, for instance, $u_r \notin X$, then $X = (X \cap Y) \cup (Y \cap \{u_1, \dots, u_{r-1}\})$ according to (9). Therefore, $U$ is generated by the set $Y$ and the elements $u_1, \dots, u_{r-1}$, which contradicts the condition that the number $r$ is minimal possible. Thus, it follows from (9) that $X = (X \cap Y) \cup \{u_1, \dots, u_r\}$, and therefore $\{u_1, \dots, u_r\} = X \setminus (X \cap Y)$, which proves the proposition.

Note that distinct $F$-subsets can have equal cobases. For instance, in the free groupoid with base $\{x, y\}$, all $F$-subsets of rank one such that the base element of this subset differs from $x$ and $y$ have the set $\{x, y\}$ as the cobase.

### 3.4. *Planar Subsets*

Let $A = (U, F)$ be a tame $A$-system. Introduce the family $\mathcal{L}$ of all finite-rank $F$-subsets. Later on, we shall see that this family forms a semimodular lattice.

**Definition.** An $F$-subset $U' \subset U$ of finite rank $r$ is said to be a *planar subset* if there exists no $F$-subset strictly containing $U'$ that is of rank $r_1 \leqslant r$.

The following assertion results from the definition.

**Proposition 3.4.** *If $U' = U[Y]$ is a planar subset of rank $r$, then*

(1) *$r(U'') > r(U')$ for any $F$-subset $U''$ strictly containing $U'$;*
(2) *$r(U'') < r(U')$ for any planar subset $U''$ strictly contained in $U'$;*
(3) *the $F$-subset $U[Y']$ is planar for any subset $Y' \subset Y$.*

**Proposition 3.5.** *For any $F$-subset $U'$ of finite rank $r$, there exists a planar subset of rank $r_1 \leqslant r$ containing $U'$.*

The assertion immediately follows from Theorem 2.1 concerning the ascending chain condition for $F$-subsets.

By the *reduced rank* of a finite-rank $F$-subset $U'$ we mean the least positive integer $r$ for which there exists a planar subset of rank $r$ containing $U'$. Denote the reduced rank of $U'$ by $p(U')$.

The following assertions result from the definition:

(1) $p(U') \leqslant r(U')$;
(2) if $U'$ is a planar subset, then $p(U') = r(U')$;
(3) $r(U'') \geqslant p(U')$ for any $F$-subset $U'' \supset U'$.

**Lemma 3.1.** *Let $U_1 = U[Y]$ and $U_2 = U[Z]$ be planar subsets of ranks $k_1$ and $k_2$, respectively, where $k_1 \leqslant k_2$, and let $n$ be the rank of their intersection $V = U_1 \cap U_2$. Then*

(1) $n \leqslant k_1$;
(2) *if* $n = k_1$, *then* $V = U_1$, *i.e.*, $U_1 \subset U_2$;
(3) *if* $n = k_1 = k_2$, *then* $V = U_1 = U_2$.

**Proof.** Consider the $F$-subset $B = U(Y \cup Z)$. We have $B \supset U_2$, and hence $r(B) \leqslant k_1 + k_2 - n$ by Theorem 3.3. If $n > k_1$, then $r(B) < k_2$, and $B$ strictly contains the planar subset $U_2$ of rank $k_2$, which is impossible. Thus, $n \leqslant k_1$. If $n = k_1$, then $r(B) \leqslant k_2$. Since $B \supset U_2$ and $r(U_2) = k_2$, the inequality is possible only if $B = U_2$, i.e., if $U_1 \subset U_2$. In particular, if $k_1 = k_2$, then it follows from the inclusion $U_1 \subset U_2$ that $U_1 = U_2$.

**Theorem 3.4.** *For any $F$-subset $U' \subset U$ of finite rank $r$, there exists a planar subset $P$ containing $U'$ and having the rank $p(U')$, where $p(U')$ is the reduced rank of $U'$, and such a set $P$ is unique.*

**Proof.** Let $P_1$ and $P_2$ be planar subsets of rank $p(U')$ that contain $U'$, and let $V = P_1 \cap P_2$. Then $V \supset U'$, and therefore $r(V) \geqslant p(U')$. However, $r(P_1) = r(P_2) = p(U')$, and therefore the relation $P_1 = P_2$ follows from Lemma 3.1.

**Definition.** The planar subset $P$ defined by Theorem 3.4 is called the *planar envelope* of the $F$-subset $U'$, or, in other words, the *planar subset generated by* $U'$.

### 3.6. *Properties of Planar Subsets*

**Theorem 3.5.** *If a planar subset $P$ contains an $F$-subset $U'$, then $P$ contains the planar envelope $P'$ of $U'$.*

**Proof.** We have $r(P') = p(U')$ and $r(P) \geqslant p(U')$. Let $V = P' \cap P$. Then $V \supset U'$. Hence, $r(V) \geqslant p(U')$. By Lemma 3.1, $r(V) \leqslant r(U')$. Therefore, $r(V) = p(U')$. Since $r(P' \cap P) = r(P')$, it follows from Lemma 3.1 that $P' \subset P$, as was to be proved.

**Theorem 3.6.** *Each intersection $V = \cap_\alpha P_\alpha$ of planar subsets $P_\alpha$ of ranks $r_\alpha$ is a planar subset of rank $r \leqslant \min_\alpha r_\alpha$. In particular, if $r = r_\alpha$ for some $\alpha$, then $V = P_\alpha$.*

**Proof.** By Theorem 3.4, there exists a unique planar subset $P \supset V$ generated by $V$. By Theorem 3.5, $P \subset P_\alpha$ for any $\alpha$, and so $P \subset V$. Hence, $P = V$ and $r(V) = r(P) = p(V) \leqslant \min_\alpha r_\alpha$. If $r = r_\alpha$ for some $\alpha$, then $V = P_\alpha$ by Lemma 3.1.

**Theorem 3.7.** *If $P_1$, $P_2$, and $P_3$ are $F$-subsets such that $P_1$ is a planar subset of $P_2$ and $P_2$ is a planar subset of $P_3$, then $P_1$ is a planar subset of $P_3$.*

**Proof.** It follows from the condition of the theorem that $r(P_1) \leqslant r(P_2) \leqslant r(P_3)$. Let $P$ be the planar subset of $P_3$ generated by $P_1$. Then $r(P) \leqslant r(P_1)$. By the previous theorem, $P_2 \cap P$ is a planar subset of $P_3$. Since $r(P) \leqslant r(P_2)$, it follows that $r(P_2 \cap P) \leqslant r(P) \leqslant r(P_1)$. Thus, together with the definition of planar subset, implies that $P_2 \cap P = P_1$, and therefore $r(P_2 \cap P) = r(P_1)$. Then $r(P_2 \cap P) = r(P)$ and $P \subset P_2$ by Theorem 3.6. Thus, $P = P_1$.

Let us give a sufficient condition for an $F$-subset $U[Y] \subset U$ of finite rank $n$ in a tame $A$-system $A = (U, F)$ with base $X \subset U$ to be planar.

**Proposition 3.6.** *If the subsets $X_y$, $y \in Y$, are pairwise disjoint and if $l(y) = \#X_y$ for all $y \in Y$, then the $F$-subset $U[Y]$ is planar.*

**Proof.** Suppose the contrary. Let $U[Y]$ be strictly contained in an $F$-subset $U[Z]$ of rank $r \leqslant n$. Consider the subset $Z_y \subset Z$, $y \in Y$. We have

$$X_y = \bigcup_{z \in Z_y} X_z;$$

hence, if at least two subsets $Z_y$ have nonempty intersection, then the corresponding subsets $X_y$ also have nonempty intersection, which contradicts the assumption. Thus, the subsets $Z_y$ are pairwise disjoint. Since $r \leqslant n$, this can occur only if $r = n$. In this case, all subsets $Z_y$ are singletons, i.e., $Z_y = \{z_y\}$, where $z_y \in Z$, and the mapping $y \mapsto z_y$ is a bijection of $Y$ onto $Z$. Note that $z_y \neq y$ for at least one $y \in Y$ because otherwise the $F$-subsets $U[Y]$ and $U[Z]$ coincide. If $z_y \neq y$, then $l(y) > l(z_y)$. On the other hand, $X_y = X_{z_y}$. Therefore, since $l(z_y) \geqslant \#X_{z_y}$, it follows that $l(y) > \#X_y$, which contradicts the condition.

## 3.7. *Lattices of Planar Subsets*

Denote by $\mathcal{L} = \mathcal{L}(U, F)$ the family of planar subsets of a tame $A$-system $A = (U, F)$. Since the intersection of planar subsets is a planar subset, it follows that $\mathcal{L}$ is equipped with the structure of a lattice by inclusion: the product $U_1 \wedge U_2$ of planar subsets $U_1 = U[Y_1]$ and $U_2 = U[Y_2]$ is defined as in the lattice $L$, and the sum $U_1 \vee U_2$ is the planar subset generated by the $F$-subset $U(Y_1 \cup Y_2)$.

Since the definitions of the sum on $\mathcal{L}$ and $L$ are distinct, $\mathcal{L}$ is not a sublattice of $L$.

**Theorem 3.8.** *The ranks of any two planar subsets $U_1$ and $U_2$ and those of the planar subsets $U_1 \wedge U_2$ and $U_1 \vee U_2$ are related by inequality* (8).

This inequality follows from a similar inequality in the lattice $L$ if one takes into account that the rank of the sum of subgroupoids in the lattice $\mathcal{L}$ does not exceed the rank of their sum in the lattice $L$.

**Definition.** We say that a planar subset $V_1$ *covers* a planar subset $V_2$ if $U_1$ strictly contains $U_2$ and there exists no planar subset $V$ distinct from $V_1$ and $V_2$ and such that $V_1 \supset V \supset V_2$.

**Theorem 3.9.** *If $V \subset U$ is an arbitrary planar subset and $U \neq V$, then any planar subset $V'$ covering $V$ is generated by $V$ and by some element $u \notin V$. Conversely, any planar subset $V'$ generated by $V$ and by an element $u \notin V$ covers $V$.*

**Proof.** Let $V' \supset V$, let $x \in V' \setminus V$, and let $V''$ be a planar subset generated by $V$ and $u$. Then $V' \supset V'' \supset V$ and $V'' \neq V$. Therefore, if $V'$ covers $V$, then $V' = V''$.

Conversely, if $V'$ is a planar subset generated by $V$ and by some element $u \notin V$, then $r(V) < r(V') \leqslant r(V) + 1$, and thus $r(V') = r(V) + 1$. Hence, $V'$ covers $V$.

**Theorem 3.10.** *The lattice $\mathcal{L}$ of the planar subsets satisfies the* semimodularity condition [4]: *if planar subsets $U_1$ and $U_2$, $U_1 \neq U_2$, cover a planar subset $U_0$, then $U_1 \vee U_2$ covers both $U_1$ and $U_2$.*

**Proof.** Let $r(U) = n$. In this case, $r(U_1) = r(U_2) = n + 1$. It follows from Theorem 3.8 that $r(U_1 \vee U_2) \leqslant n + 2$. On the other hand, since $U_1 \neq U_2$, it follows that the subset $U_1 \vee U_2$ strictly contains $U_1$ and $U_2$, and therefore $r(U_1 \vee U_2) > n + 1$. Hence, $r(U_1 \vee U_2) = n + 2$.

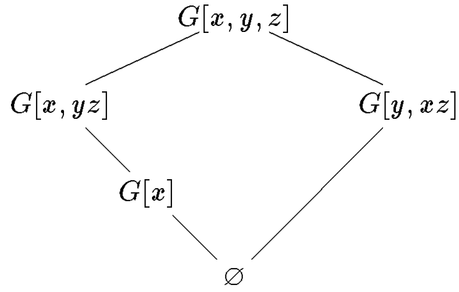


**Fig. 4**

Note that the lattice $\mathcal{L}$ can be not modular. For instance, the lattice $\mathcal{L}$ related to the free groupoid $G[x, y, z]$ contains the sublattice shown in Fig. 4. However, modular lattices have no sublattices of this form.

### 3.8. *Geometric Structures on Tame A-Systems*

Since the lattice $\mathcal{L}$ of all planar subsets is semimodular, it follows that the support $U$ of a tame $A$-system can naturally be treated as a projective space, and the planar subsets of rank $r$ by themselves can be viewed as $(r-1)$-dimensional planes in this space. In particular, the planar subsets of ranks one and two will be referred to as points and lines, respectively. This point of view is especially convenient in the case of a free idempotent $A$-system because, for this system, all planar subsets of rank one are singletons. However, as in the case of an arbitrary tame $A$-system, one can replace every planar set of rank one (i.e., a "projective point") by an element $u \in U$ generating this set.

In the geometry thus arising, the main axioms concerning the unions and intersections of planes in projective spaces are satisfied. For instance, one and only one line can be drawn to pass through two distinct points. Every line can be disjoint from a plane, can have exactly one point of intersection with this plane, or can belong to the plane. Exactly one two-dimensional plane passes through two (noncoinciding) intersecting lines, etc.

The specific feature of this geometry is that every $k$-dimensional plane is uniquely equipped with the base formed by $(k+1)$ points, i.e., the set of indecomposable elements. Moreover, the base of the intersection of two planes is contained in the union of the bases of these planes. For this reason, for instance, the family of lines passing through a chosen point $u$ is partitioned into two subsets formed by the lines that contain or do not contain the point $u$ in the set of their base points.

If points $a$, $b$, and $c$ form the base of a two-dimensional plane, then any line passing through two of these three points has these points as the base. If a line $l$ on the plane intersects the lines $\overline{a, b}$ and $\overline{a, c}$, then the points of intersection form the base of the line $l$. Hence, $l$ cannot intersect the line $\overline{b, c}$.

### 3.9. *Planes*

**Definition.** An $F$-subset $V \subset U$ of a tame $A$-system $A = (U, F)$ is called a *plane* if any planar subset in $V$ is a planar subset in $U$.

In particular, the support $U$ of the $A$-system and the empty set $\varnothing$ are planes.

The following assertions result from the definition.

(1) The notion of plane is transitive: if $V_1$ is a plane in $U$ and $V_2$ is a plane in $V_1$, then $V_2$ is a plane in $U$.
(2) Every planar subset in $U$ is a plane and, conversely, every plane of finite rank is a planar subset. In particular, if the rank of the support $U$ is finite, then every plane in $U$ is a planar subset.
(3) Every plane $V \subset U$ is the union of all planar subset in $U$ belonging to $V$.

**Theorem 3.11.** *The intersection $V = \bigcap_\alpha V_\alpha$ of any family of planes $V_\alpha$ is a plane.*

**Proof.** Let $V' \subset V$ be an arbitrary planar subset of $V$ and let $V'_\alpha$ be a planar covering of $V'$ in $V_\alpha$, i.e., a minimal planar subset of $V_\alpha$ containing $V'$. By the definition of plane, every $V'_\alpha$ is a planar subset of $U$, and hence so is the intersection

$$V'' = \bigcap_\alpha V'_\alpha.$$

Let us prove that $V' = V''$. Indeed, since $r(V'_\alpha) \leqslant r(V')$ for any $\alpha$, it follows that $r(V'') \leqslant r(V')$. Since $V'$ is a planar subset of $V$ and $V' \subset V'' \subset V$, this inequality is possible only if $V'' = V'$.

**Corollary.** *For any subset $A \subset U$, there exists the smallest plane containing this subset, namely, the intersection of all planes in $U$ containing $A$.*

**Definition.** The smallest plane in $U$ containing a subset $A \subset U$ is called the *planar envelope* or the *planar covering* of $A$. Introduce an operation $*$ on the family of subsets of $U$; by definition, $A * B$ is the planar covering of the set $A \cup B$.

The operation $*$ is commutative, associative, and also has the following properties.

(1) If $(A \cup B) \subset (A' \cup B')$, then $A * B \subset A' * B'$. In particular, if $(A \cup B) = (A' \cup B')$, then $A * B = A' * B'$; if $B \subset A$, then $A * B = A * \varnothing$.
(2) $A * A * B = A * B$ for any subsets $A$ and $B$.
(3) If $V$ is a plane, then $V * V = V$.
(4) For any planes $V$ and $W$ in $U$, we have

$$V \vee W = V * W = \cup(V' * W'),$$

where the union is taken over the family of all planar subsets $V'$ and $W'$ contained in $V$ and $W$, respectively.

Let us present a criterion for a subset $V \subset U$ to be a plane (in terms of the operation $*$).

**Proposition 3.7.** *A subset $V \subset U$ is a plane in $U$ if and only if $V$ is the union of some planar subsets in $U$ and, for any planar subsets $V'$ and $V''$ contained in $V$, the planar subset $V' * V''$ is also contained in $V$.*

**Proposition 3.8.** *For any plane $V \subset U$ and any element $x \notin V$, we have*

$$V * \{x\} = \cup(V' * \{x\}),$$

*where the union is taken over all planar subsets $V'$ in $U$ contained in $V$.*

**Proof.** Write $W = \cup(V' * \{x\})$. For any planar subsets $V'$ and $V''$ in $V$, we have

$$(V' * \{x\}) * (V'' * \{x\}) = (\widetilde{V} * \{x\}), \quad \text{where} \quad \widetilde{V} = V' * V''.$$

Since $\widetilde{V}$ is a planar subset contained in $V$, it follows that the set $W$ satisfies the conditions of Proposition 3.7, and therefore it is a plane in $U$. Since $V \subset W$ and $x \in W$, it follows that $V * \{x\} \subset W$. On the other hand, $W \subset V * \{x\}$ because $V' * \{x\} \subset V * \{x\}$ for any planar subset $V' \subset V$. Thus, $V * \{x\} = W$.

**Definition.** We say that a plane $V_1$ *covers* a plane $V_2$ if $U_1$ strictly contains $U_2$ and there exists no plane $V$ distinct from $V_1$ and $V_2$ and such that $V_1 \supset V \supset V_2$.

**Theorem 3.12.** *A plane $V'$ covers a plane $V \subset U$, $V \neq U$, if and only if $V' = V * \{x\}$, where $x \notin V$. Thus, the set planes covering a plane $V \neq U$ is not empty.*

**Proof.** In one direction the assertion is clear, namely, if a plane $V'$ strictly contains a plane $V$ and $x \in V' \setminus V$, then $V' \supset V * \{x\} \supset V$ and $V * \{x\} \neq V$. Thus, if $V'$ covers $V$, then $V' = V * \{x\}$.

Conversely, let us prove that a plane $V * \{x\}$, where $x \notin V$, covers the plane $V$. Suppose the contrary. Let there exist a plane $W$ distinct from $V$ and $V * \{x\}$ and such that $V * \{x\} \supset W \supset V$. Let $u \in W \setminus V$. Then $V * \{u\} \neq V$, $V * \{u\} \neq V * \{x\}$, and $V * \{x\} \supset V * \{u\} \supset V$.

By Proposition 3.8, there exists a planar subset $V' \subset V$ such that $u \in V' * \{x\}$, and hence $V' * \{x\} \supset V' * \{u\} \supset V'$. By Theorem 3.10, the planar set $V' * \{x\}$ covers the planar subset $V'$. Therefore, since $V' * \{u\} \neq V' * \{x\}$, it follows that $V' * \{u\} = V'$, i.e., $u \in V'$, which is not the case.

### 3.10. *Lattice of all Planes*

It follows from Theorem 3.11 that the set $\widetilde{\mathcal{L}}$ of all planes in the support $U$ of a tame $A$-system $A = (U, F)$ is equipped with the structure of a lattice with respect to embedding. In this lattice, for any planes $V_1$ and $V_2$, the element $V_1 \wedge V_2$ is defined as the intersection of these planes and $V_1 \vee V_2$ as the smallest plane containing $V_1$ and $V_2$.

The family $\mathcal{L}$ of planar subsets is a sublattice of this lattice. Here are other examples of sublattices in $\widetilde{\mathcal{L}}$:

1) the sublattice of planes of countable rank,

2) the sublattice of planes of finite corank.

**Remark.** The family of planes $V \subset U$ such that $V * \{u_1, \dots, u_n\} = U$ for an appropriate finite subset $\{u_1, \dots, u_n\} \subset U$ is closed with respect to union but not closed with respect to intersection; therefore, it does not form a sublattice of $\widetilde{\mathcal{L}}$. For instance, in the free groupoid with the base $\{x, y, z_1, \dots, z_n, \dots\}$, the $F$-subsets $V_1$ and $V_2$ with the bases $\{x, yz_1, \dots, yz_n, \dots\}$ and $\{y, xz_1, \dots, xz_n, \dots\}$ are planes. The relations $V_1 * \{y\} = V_2 * \{x\} = U$ hold; however, $V_1 \cap V_2 = \varnothing$. We stress that $V_1$ and $V_2$ have infinite corank and that their intersection is the empty set.

**Theorem 3.13.** *The lattice $\widetilde{\mathcal{L}}$ is semimodular.*

**Proof.** It suffices to prove that if planes $U_1$ and $U_2$, $U_1 \neq U_2$, cover the plane $U_0$, then $U_1 \vee U_2$ covers both $U_1$ and $U_2$.

It follows from Theorem 3.12 that there exist elements $x_1, x_2 \notin V$ such that $V_1 = V * \{x_1\}$ and $V_2 = V * \{x_2\}$, where $x_1 \neq V * \{x_2\}$ and $x_2 \neq V * \{x_1\}$. Then the plane $V_1 \vee V_2 = V * \{x_1\} * \{x_2\}$ covers $V_1$ and $V_2$ because $V * \{x_1\} * \{x_2\} = V_1 * \{x_2\} = V_2 * \{x_1\}$.

### 3.11. *Inductive Limits of Tame A-Systems and Related Lattices of Planes*

Let us extend the class of $A$-systems passing from the tame $A$-systems to their inductive limits. Assume that a family of tame $A$-systems $A^\alpha = (U^\alpha, F^\alpha)$ is given, where the index $\alpha$ ranges over a partially ordered set $\Lambda$ in which, for any $\alpha, \beta \in \Lambda$, there exists a $\gamma \in \Lambda$ for which $\alpha < \gamma$ and $\beta < \gamma$.

Further, we assume that the following objects are defined for any ordered pair of indices $\alpha, \beta$, where $\alpha < \beta$:

1) an injection (embedding)

$$\sigma_{\beta\alpha}\colon U^\alpha \hookrightarrow U^\beta,$$

2) bijections

$$\tau_{\beta\alpha}\colon F_n^\alpha \to F_n^\beta, \qquad n = 1, 2, \dots,$$

where $F_n^\alpha$ is the subset of the $n$-ary operations. It is assumed that

$$\sigma_{\gamma\beta} \circ \sigma_{\beta\alpha} = \sigma_{\gamma\alpha}, \qquad \tau_{\gamma\beta} \circ \tau_{\beta\alpha} = \tau_{\gamma\alpha}$$

for any ordered triple of indices $\alpha < \beta < \gamma$.

By 2), one can assume that any set $F^\alpha$ is identified with a chosen set $F$.

Sets $\{\sigma_{\beta\alpha}\}$ and $\{\tau_{\beta\alpha}\}$ are said to be *compatible* if the following properties hold for any ordered pair $\alpha, \beta$, where $\alpha < \beta$:

a) $\sigma_{\beta\alpha} U^\alpha \subset U^\beta$ is an $F^\beta$-subset;

b) $\sigma_{\beta\alpha}(f(u_1, \dots, u_n)) = (\tau_{\beta\alpha} f)(\sigma_{\beta\alpha} u_1, \dots, \sigma_{\beta\alpha} u_n)$ for any $n = 1, 2, \dots$, any operation $f \in F_n^\alpha$, and any elements $u_1, \dots, u_n \in U^\alpha$.

To any compatible system $\{\sigma_{\beta\alpha}\}$ and $\{\tau_{\beta\alpha}\}$ one can assign the inductive limit $A = \lim\operatorname{ind}_\alpha A^\alpha$, which is an $A$-system $A = (U, F)$ with $U = \lim\operatorname{ind}_\alpha U^\alpha$.

**Definition.** An $A$-system $A = \lim\operatorname{ind}_\alpha A^\alpha$ is said to be *weakly tame* if, for any ordered pair $\alpha, \beta$, where $\alpha \leqslant \beta$, the subset $\sigma_{\beta\alpha} U^\alpha \subset U^\beta$ is a plane in $U^\beta$.

Obviously, every tame $A$-system is weakly tame.

For example, the inductive limit of a sequence groupoids $G[X_n]$, $n = 1, 2, \dots$, with the bases $X_n = \{x_{n,1}, \dots, x_{n,2^{n-1}}\}$ is a weakly tame $A$-system. The embedding $G[X_n] \hookrightarrow G[X_{n+1}]$ is given by the formula

$$x_{n,i} = x_{n+1,2i-1}\, x_{n+1,2i}, \qquad i = 1, \dots, 2^{n-1}.$$

**Definition.** Let $A = (U, F)$, where $U = \lim\operatorname{ind}_\alpha U^\alpha$, be an arbitrary weakly tame $A$-system. A subset $V \subset U$ is called a *plane* if the intersection $V^\alpha = V \cap U^\alpha$ is a plane in $U^\alpha$ for any index $\alpha$.

According to this definition, every plane $V \subset U$ is the inductive limit of the planes $V^\alpha = V \cap U^\alpha \subset U^\alpha$, $V = \lim\operatorname{ind}_\alpha V^\alpha$.

Obviously, any plane is an $F$-subset of $U$.

It follows from Theorem 3.11 that the intersection of each family of planes of a weakly tame $A$-system is also a plane. Thus, the set of planes of a weakly tame $A$-system is equipped with the structure of a lattice with respect to the operation of embedding.

The notion of planar coverings of subsets and the operation $*$ can naturally be extended to the weakly tame $A$-systems. Note that, for every plane $V = \lim\operatorname{ind}_\alpha V^\alpha$ and any point $x \notin V$, we have the relation

$$V * \{x\} = \lim_\alpha \operatorname{ind} V^\alpha * \{x\}.$$

**Definition.** As in the case of tame $A$-systems, we say that a plane $V_1$ of a weakly tame $A$-system covers a plane $V_2$ if $U_1$ strictly contains $U_2$ and there exists no plane $V$ distinct from $V_1$ and $V_2$ and such that $V_1 \supset V \supset V_2$.

**Theorem 3.14.** *A plane $V'$ of a weakly tame $A$-system $A = (U, F)$, where $U = \lim\operatorname{ind}_\alpha U^\alpha$, covers a plane $V \subset U$, $V \neq U$, if and only if $V' = V * \{x\}$, where $x \notin V$. Thus, the set of planes covering a plane $V \neq U$ is not empty.*

**Proof.** If the plane $V'$ strictly contains the plane $V$ and $x \in V' \setminus V$, then $V' \supset V * \{x\} \supset V$ and $V * \{x\} \neq V$. Thus, if $V'$ covers $V$, then $V' = V * \{x\}$.

Conversely, let us prove that the plane $V * \{x\}$, where $x \notin V$, covers the plane $V$. Assume the contrary. Let there exist a plane $W$ distinct from $V$ and $V * \{x\}$ and such that $V * \{x\} \supset W \supset V$. Let $u \in W \setminus U$. Then $V * \{x\} \supset V * \{u\} \supset V$, where the planes $V * \{x\}$, $V * \{u\}$, and $V$ are pairwise distinct.

Since $V * \{x\} = \lim\operatorname{ind}_\alpha V^\alpha * \{x\}$, where $V^\alpha = V \cap U^\alpha$, it follows that there is an index $\alpha_0$ for which $x \in U^\alpha$ and $u \in V^\alpha\{x\}$ for any $\alpha > \alpha_0$. Thus, $V^\alpha * \{x\} \supset V^\alpha * \{u\} \supset V^\alpha$ for any $\alpha > \alpha_0$. By Theorem 3.12, the plane $V^\alpha * \{x\}$ covers $V^\alpha$. Since $V^\alpha * \{u\} \neq V^\alpha$, this implies that $V^\alpha * \{x\} = V^\alpha * \{u\}$ for any $\alpha > \alpha_0$. Then $V * \{x\} = V * \{u\}$, which contradicts the assumption.

**Theorem 3.15.** *The lattice of planes of a weakly tame $A$-system is semimodular.*

The proof is just the same as in Theorem 3.13.

## 4. TOPOLOGICAL STRUCTURES ON $A$-SYSTEMS AND ON THE SETS OF THEIR SUBSYSTEMS

### 4.1. *Topological A-Systems*

Let us define two types of topological $A$-systems, namely, $A$-systems $A = (U, F)$ for which the topology is defined on the support $U$ only and $A$-systems for which both the support $U$ and the fundamental set $F$ are equipped with some topologies.

**Definition.** An $A$-system $A = (U, F)$ is called an *AT-system* if the support $U$ is equipped with the structure of a topological space with respect to which the mappings

$$(u_1, \dots, u_n) \in U^{\times n} \to f(u_1, \dots, u_n) \in U$$

are continuous for any $n$ and any $f \in F_n$.

**Definition.** An $A$-system $A = (U, F)$ is called an *ATF-system* if $U$ and $F$ are equipped with the structures of topological spaces such that the mappings

$$(f, u_1, \dots, u_n) \in F_n \times U^{\times n} \to f(u_1, \dots, u_n) \in U$$

are continuous with respect to the topologies for any $n$.

If the topology on $F$ is discrete, then these definitions are equivalent.

Every topology on the support $U$ of the $A$-system $A = (U, F)$ induces some topology on the family $L = L(U, F)$ of all finitely generated subsystems of this $A$-system or, equivalently, on the family of finitely generated $F$-subsets $U' \subset U$. This topology is defined as follows.

Denote by $M(V_{y_1}, \dots, V_{y_n})$, where $V_{y_i} \subset U$ are some neighborhoods of the points $y_1, \dots, y_n$, respectively, the family of all $F$-subsets $U(z_1, \dots, z_n)$ generated by the elements $z_i \in V_{y_i}$, $i = 1, \dots, n$.

**Definition.** Let us introduce a topology on $L = L(U, F)$ as follows. For a base of neighborhoods of any finitely generated $F$-subset $U' \subset U$, we take the family of sets $M(V_{y_1}, \dots, V_{y_n})$, where the collection $Y = \{y_1, \dots, y_n\}$ ranges over the bases in $U' \subset U$ and $V_{y_1}, \dots, V_{y_n}$ range over the bases of neighborhoods of the elements in $Y$.

This definition can be simplified for $N$-systems because, for an $N$-system, every finitely generated $F$-subset $U' \subset U$ has a unique base $Y = \{y_1, \dots, y_n\}$. This implies the following assertion.

**Proposition 4.1.** *A Hausdorff topology on the support $U$ of an $N$-system induces a Hausdorff topology on the families $L_n \subset U$ of all $F$-subsets $U' \subset U$ of arbitrarily chosen rank $n$.*

### 4.2. *Free AT-Systems*

Let $A[X] = (U, F)$ be a free $A$-system, and let the base $X \subset U$ of this system be equipped with the structure of a topological space, i.e., a base of neighborhoods of any point $x \in X$ is defined. Let us construction the extension of the topology on $X$ to a topology on $U$ with respect to which all operations $f \in F$ are continuous.

To this end, let us define the bases of neighborhoods of all points $u \in U$ by induction on the height $h(u)$. If $h(u) = 1$, i.e., $u \in X$, then the base of neighborhoods $V_u \subset X$ of the point $u$ is (originally) defined. Let the bases of neighborhoods be already defined for all points of height less than $n$, and let $h(u) = n > 1$. Then $u$ can be represented (and this representation is unique) in the form

$$u = f(u_1, \dots, u_m),$$

where $h(u_i) < n$, $i = 1, \dots, m$, and hence the bases of neighborhoods of the elements $u_i$ have already been defined. Let us define the base of neighborhoods of the element $u$ as the family of sets

$$f(V_{u_1}, \dots, V_{u_m}) = \{u' = f(u'_1, \dots, u'_m) \mid u'_1 \in V_{u_1}, \dots, u'_m \in V_{u_m}\},$$

where $V_{u_1}, \dots, V_{u_m}$ range over the bases of neighborhoods of the elements $u_1, \dots, u_m$.

It follows from the definition that all operations $f \in F$ are continuous with respect to the topology thus introduced. According to Subsection 4.1, we refer to an $A$-system $A[X] = (U, F)$ with the topology on $U$ thus defined as a *free AT-system*.

In what follows, it is assumed that the topology on $X$ is Hausdorff. Then the topology on $U$ is also Hausdorff.

It follows from the definition of the topology on $U$ that the neighborhoods $f(V_{u_1}, \dots, V_{u_m})$ of the element $u = f(u_1, \dots, u_m)$ are homeomorphic to the Cartesian product $V_{u_1} \times \dots \times V_{u_m}$. Hence, applying the usual induction on the height of elements, we obtain the following assertion.

**Proposition 4.2.** *The properties of local connectedness and local compactness are preserved under the extension of the topology from $X$ to $U$.*

**Proposition 4.3.** *Any $S$-subset $U_S \subset U$ (i.e., a subset with an arbitrarily fixed decomposition scheme $S$) is open and closed.*

**Proof.** It suffices to prove (by induction on the height) that, for any $u \in U$, there exists a neighborhood all of whose elements have the same decomposition scheme as that of $u$. In the case of $h(u) = 1$, the assertion is obvious. If $h(u) = n > 1$, then let us represent $u$ in the form $u = f(u_1, \dots, u_m)$, where $h(u_i) < n$. By the induction assumption, for any $u_i$, there exists a neighborhood $V_{u_i}$ all of whose elements have the same decomposition scheme as that of $u_i$. Then all elements $u'$ in the neighborhood $f(V_{u_1}, \dots, V_{u_m})$ of $u$ have the same decomposition scheme as that of $u$.

**Corollary 4.1.** *The subsets of elements $u \in U$ of any fixed height and the subsets of elements of any fixed length open and closed.*

**Corollary 4.2.** *All $F$-subsets of finite rank are discrete.*

(This holds because these sets contain only finitely many elements of any chosen height.)

**Proposition 4.4.** *Let $A[X] = (U, F)$ be a free $AT$-system with base $X \subset U$, let $y_1, \dots, y_n$ be arbitrary elements in $U$ (not necessarily pairwise distinct), and let $V_{y_1}, \dots, V_{y_n}$ be any neighborhoods of these elements in $U$. Then, if $X$ contains no isolated points, then there exist elements $z_i \in V_{y_i}$, $i = 1, \dots, n$, such that the sets $X_{z_i}$ are pairwise disjoint and $l(z_i) = \#X_{z_i}$, $i = 1, \dots, n$.*

**Proof.** Let us proceed by induction on $N = \max(h(y_1), \dots, h(y_n))$. For $N = 1$, i.e., if $y_i \in X$, $i = 1, \dots, n$, the assertion is obvious. Let us prove this fact for an arbitrary $N > 1$ assuming that the assertion is already proved for the positive integers less than $N$.

If $\max(h(y_1), \dots, h(y_n)) = N$, then, to be definite, set $h(y_i) = N$ for $i \leqslant k$ and $h(y_i) < N$ for $i > k$. In this case, any element $y_i$, $i \leqslant k$, can be represented in the form $y_i = f_i(y_{i1}, \dots, y_{is_i})$, where $h(y_{ij}) < N$. Let $V_{ij}$ be neighborhoods of elements $y_{ij}$, and let $V_j$, $j > k$, be neighborhoods of elements $y_j$, $j > k$. By the induction assumption, there exist elements $z_{ij} \in V_{ij}$ and $z_j \in V_j$ such that the sets $X_{z_{ij}}$ and $X_{z_j}$ are pairwise disjoint and $l(z_{ij}) = \#X_{z_{ij}}$ and $l(z_j) = \#X_{z_j}$. Then the elements $z_i = f_i(z_{i1}, \dots, z_{is_i})$, $i \leqslant k$, and $z_{k+1}, \dots, z_n$ have the desired property.

## 4.3. *Compatible Systems of Neighborhoods*

Let $U' \subset U$ be an $F$-subset of finite rank of a free $AT$-system $A = (U, F)$. To any point $u \in U'$ we assign a neighborhood $V_u \subset U$ of $u$.

**Definition.** A system of neighborhoods $\{V_u \mid u \in U'\}$ is said to be *compatible with an $F$-subset* $U' \subset U$ if

1) every neighborhood $V_u$ is contained in some $S$-subset,
2) the neighborhoods $V_u$ are pairwise disjoint,
3) if $u = f(u_1, \dots, u_k)$, where $u_i \in U'$, then $V_u = f(V_{u_1}, \dots, V_{u_k})$.

Note that, if $Y = \{y_1, \dots, y_n\}$ is a base of an $F$-subset $U'$, then any neighborhood system compatible with $U'$ is uniquely defined by some neighborhoods $V_{y_i}$, $i = 1, \dots, n$, satisfying conditions 1) and 2).

The following two assertions immediately result from the definition of compatible systems.

**Proposition 4.5.** *The union of neighborhoods entering any system compatible with a finite-rank $F$-subset $U' \subset U$ is an $F$-subset in $U$.*

**Proposition 4.6.** *If $\{V_u \mid u \in U'\}$ is a system of neighborhoods compatible with an $F$-subset $U'$, then, for any $F$-subset $U'' \subset U'$, the system of neighborhoods $\{V_u \mid u \in U''\}$ is compatible with $U''$.*

**Theorem 4.1** (separation of $F$-subsets of finite rank). *Any disjoint $F$-subsets $U_1$ and $U_2$ of finite rank in a free $AT$-system are contained in some disjoint open $F$-subsets $\widetilde{U}_1$ and $\widetilde{U}_2$.*

**Proof.** Introduce the $F$-subset $U' = U_1 \vee U_2$ of finite rank and define an arbitrary system of neighborhoods $\{V_u \mid u \in U'\}$ compatible with $U'$. Set

$$\widetilde{U}_1 = \bigcup_{u \in U_1} V_u, \qquad \widetilde{U}_2 = \bigcup_{u \in U_2} V_u.$$

It follows from Propositions 4.5 and 4.6 that $\widetilde{U}_1$ and $\widetilde{U}_2$ are disjoint open $F$-subsets containing $U_1$ and $U_2$, respectively.

### 4.4. *Topological Space of F-Subsets of a Free AT-System*

Denote by $M = M[X]$ the set of all $F$-subsets of finite rank of a free $AT$-system $A[X] = (U, F)$ with base $X \subset U$. According to Subsection 4.1, the set $M$ can be equipped with a Hausdorff topology induced by the Hausdorff topology on $U$. Namely, the base of neighborhoods of any $F$-subset $U[Y] \subset U$ with base $Y = \{y_1, \dots, y_n\}$ consists of the sets $M(V_{y_1}, \dots, V_{y_n}) \subset M$, where $V_{y_i}$ ranges over a base of neighborhoods of the point $y_i$ for any $i = 1, \dots, n$.

Let $U_0 = U[y_1, \dots, y_n]$ be an arbitrary $F$-subset of finite rank of the free $AT$-system $A[X] = (U, F)$, and let $V_{y_1}, \dots, V_{y_n}$ be neighborhoods of $y_1, \dots, y_n$ satisfying conditions 1) and 2) in Subsection 4.3.

**Theorem 4.2.** *For any $F$-subset $U' \subset U$ belonging to a neighborhood $M(V_{y_1}, \dots, V_{y_n})$ of an $F$-subset $U_0$, there is a natural isomorphism of the $F$-subsets, $U' \to U_0$.*

**Proof.** It follows from the definition of the neighborhood $M(V_{y_1}, \dots, V_{y_n})$ that we have $U' = U(z_1, \dots, z_n)$, where $z_i \in V_{y_i}$, $i = 1, \dots, n$. Let $\{V_u \mid u \in U_0\}$ be the compatible system of neighborhoods generated by the neighborhoods $V_{y_i}$. By the compatibility, every element $z \in U'$ belongs to one of the neighborhoods $V_u$. Since the neighborhoods $V_u$ are pairwise disjoint, this relation defines a mapping $U' \to U_0$ taking $z_i$ to $y_i$, $i = 1, \dots, n$, and preserving the multiplication. Since $\{y_1, \dots, y_n\}$ is a base set, it follows that $\{z_1, \dots, z_n\}$ is also a base set, and therefore the mapping $U' \to U_0$ is an isomorphism of $F$-subsets.

**Corollary 4.1.** *The subsets $M_n \subset M$ of $F$-subsets of any chosen rank $n$ are open and closed in $M$.*

**Corollary 4.2.** *Any neighborhood of the form $M(V_{y_1}, \dots, V_{y_n})$ in the space of $F$-subsets $M$ is homeomorphic to the Cartesian product of the corresponding neighborhoods, $V_{y_1} \times \dots \times V_{y_n}$.*

### 4.5. *Theorem on the Planar Subsets of a Free AT-System*

**Theorem 4.3.** *If the base $X$ of a free $AT$-system contains no isolated points, then the subset $M^0 \subset M$ of planar subsets is open and dense in $M$.*

**Proof.** Let $M(V_{y_1}, \dots, V_{y_n})$ be a neighborhood of an arbitrary $F$-subset $U' = U[y_1, \dots, y_n] \in U$. By Proposition 4.4, there exist elements $z_i \in V_{y_i}$, $i = 1, \dots, n$, such that the sets $X_{z_i}$ are pairwise disjoint and $l(z_i) = \#X_{z_i}$, $i = 1, \dots, n$. In this case, by Proposition 3.6, the $F$-subset $U(z_1, \dots, z_n)$ contained in the neighborhood $M(V_{y_1}, \dots, V_{y_n})$ is planar. Hence, the family $M^0$ of planar subsets is dense in $M$.

Let us prove that $M^0$ is open. Let $U[Y]$, where $Y = \{y_1, \dots, y_n\}$, be an arbitrary planar subset of $U = U[X]$. Introduce the $F$-subset $U[Z] \supset U[Y]$, where

$$Z = \bigcup_{i=1}^{n} X_{y_i};$$

this set is planar because $Z \subset X$. Choose an arbitrary compatible system of neighborhoods $\{V_u \mid u \in U[Z]\}$ for $U[Z]$. Since $U[Y] \subset U[Z]$, it follows that the set $U[Y]$ contains all elements of a compatible system of neighborhoods $\{V_u \mid u \in U[Y]\}$ for $U[Y]$.

Consider the neighborhood $M(V_{y_1}, \dots, V_{y_n})$ of the planar subset $U[Y]$. Let us prove that all $F$-subsets in this neighborhood are planar.

Let $U[Y']$, where $Y' = \{y'_1, \dots, y'_n\}$, be an arbitrary $F$-subset in this neighborhood, i.e., $y'_i \in V_{y_i}$, $i = 1, \dots, n$. By Theorem 4.2, the elements $y'_i$ form a base in $U[Y']$. We claim that the $F$-subset $U[Y']$ is planar.

Introduce the $F$-subset $U[Z'] \supset U[Y']$, where

$$Z' = \bigcup_{i=1}^{n} X_{y'_i}.$$

By Theorem 4.2, there is an isomorphism of $F$-subsets of the form $\sigma\colon U[Y'] \to U[Y]$. Let us extend this isomorphism to a mapping $\sigma\colon U[Z'] \to U[Z]$. By construction, every element $u' \in U[Z']$ belongs to one of the neighborhoods $V_u$, $u \in U[Z]$. Since these neighborhoods are pairwise disjoint, this defines a mapping $\sigma\colon U[Z'] \to U[Z]$ that coincides on $U[Y']$ with the original mapping. The extended mapping $\sigma$ is surjective and preserves the operation of multiplication, the schemes, and the relation of subordination for the elements. However, $\sigma$ is not a bijection in general.

Suppose that the $F$-subset $U[Y']$ is not planar. Then the set $U[Y']$ is strictly contained in an $F$-subset $U[W] \subset U[Z']$ with base $W = \{w_1, \dots, w_r\}$, where $r \leqslant n$ and

$$W = \bigcup_{i=1}^{n} W_{y'_i}.$$

Note that $U = \{u_1, \dots, u_r\}$, where $r \leqslant n$ and

$$\bigcup_{i=1}^{n} U_{y_i} = U.$$

We have $\sigma\, U[W] \supset U[Y]$ and $r(\sigma\, U[W]) \leqslant r$. Since among the elements $w_i$ there exist elements that are strictly subordinated to at least one element $y'_i$ and the mapping $\sigma$ preserves the subordination relation, it follows that $\sigma\, U[W] \neq U[Y]$. This contradicts the condition that $U[Y]$ is a planar subset.

### 4.6. *Free Commutative and Free Idempotent AT-Systems*

The topologization of free $A$-systems $A[X] = (U, F)$ presented in Subsection 4.2 can also be extended to all free commutative and free idempotent $A$-systems. Thus, two new classes of topological $A$-systems arise, namely, the free commutative and free idempotent $AT$-systems.

The above assertions and their proofs for free $AT$-systems, except for Proposition 4.5 and Theorem 4.1, remain valid for these new classes of $AT$-systems. In particular, the topology on these $AT$-systems is Hausdorff.

Proposition 4.5 and Theorem 4.1 hold for the free commutative $AT$-systems but are generally wrong for free idempotent $AT$-systems. The reason is that only expressions of the form $u = f(u_1, \dots, u_n)$, where there are at least two distinct elements of the form $u_i$, enter the definition of compatible systems of neighborhoods. Therefore, sets of the form $f(V_x, \dots, V_x)$ do not enter any compatible system of neighborhoods.

**Remark.** The topologization presented here can be applied to an arbitrary tame $A$-system. However, in the general case, the resulting topology on $U$ can be non-Hausdorff, as can be seen from the following arguments.

Let $U = U[X]$ be a groupoid with base $X$. Suppose that elements $x_1, x_2 \in X$, $x_1 \neq x_2$, do not commute and any neighborhoods $V_{x_1}$ and $V_{x_2}$ of $x_1$ and $x_2$, respectively, contain commuting elements $y_1 \in V_{x_1}$ and $y_2 \in V_{x_2}$. Then the neighborhoods $V_{x_1 x_2} = V_{x_1} V_{x_2}$ and $V_{x_2 x_1} = V_{x_2} V_{x_1}$ of the elements $x_1 x_2$ and $x_2 x_1$ have nonempty intersection.

An example of such a groupoid can be constructed indeed.

### 4.7. *Free ATF-Systems*

Suppose that the structure of topological (Hausdorff) space is given not only on the base $X \subset U$ but also on the subsets $F_n \subset F$ of the $n$-ary operations of a free $A$-system $A[X] = (U, F)$. Let us construct a topology on $U$ induced by the topologies on $X$ and $F_n$, $n = 1, 2, \dots$ We shall now define bases of neighborhoods of the elements $u \in U$ by induction on their height $h(u)$.

For the elements of unit height, i.e., for the elements $u \in X$, bases of neighborhoods are already (originally) defined. If $h(u) = n > 1$, then the element $u$ can uniquely be represented in the form $u = f(u_1, \dots, u_n)$, where $h(u_i) < n$, $i = 1, \dots, n$. Let us define a base of neighborhoods of the element $u$ as the family of sets

$$V_f(V_{u_1}, \dots, V_{u_n}) = \{u' = f'(u'_1, \dots, u'_n) \mid f' \in V_f,\, u'_i \in V_{u_i},\, i = 1, \dots, n\},$$

where $V_f$ and $V_{u_1}, \dots, V_{u_n}$ range over bases of neighborhoods of the operation $f \in F_n$ and of the elements $u_i \in U$, respectively.

It follows from the construction that the system $A[X] = (U, F)$ equipped with the topology on $U$ thus defined is an $ATF$-system. We call it a *free ATF-system.*

If the topology on $F$ is discrete, then the $ATF$-topology coincides with the $AT$-topology. If the topology on $F$ is not discrete, then the $ATF$-topology is weaker than the $AT$-topology, i.e., the identity mapping $U \to U$ of the space $U$ with $AT$-topology onto the space $U$ equipped with the $ATF$-topology is continuous but not homeomorphic [3].

In the $ATF$-topology, the subsets of elements having an arbitrarily fixed decomposition scheme are closed but not open. The subsets of elements whose decomposition schemes differ only on the labels at the vertices are open and closed.

**Note.** One can also introduce the $ATF$-topology starting from the embedding $U \to G[X \cup F]$ of the set $U$ in the free groupoid $G[X \cup F]$ generated by the set $X \cup F$. Namely, $G[X \cup F]$ can be equipped with the $AT$-topology induced by the given topology on $X \cup F$. The corresponding induced topology on the image of $U$ coincides with the $ATF$ topology on $U$.

Consider the space of the finite-rank $M[X]$-subsets $U' \subset U$ of a free $ATF$-system $A[X] = (U, F)$ with the topology induced by the topology on $U$.

**Proposition 4.7.** *The subspace $M^0[X] \subset M[X]$ of planar subsets is dense in $M[X]$.*

Indeed, the $ATF$-topology on $U$ is weaker than the $AT$-topology. Hence, the topology on $M[X]$ induced by the $ATF$-topology on $U$ is weaker than the topology induced on $U$ by the $AT$-topology. Therefore, the assertion follows from the similar assertion for the $AT$-topology (Theorem 4.3).

The above topologization of the free $A$-systems $A[X] = (U, F)$ can also be used for the free commutative and free idempotent $A$-systems. Thus, two new classes of topological $A$-systems arise, namely, the free commutative and free idempotent $ATF$-systems. Proposition 4.7 remains valid for these systems as well.

### 4.8. *Secondary Topology on the Free Idempotent A-Systems*

Let $A[X] = (U, F)$ be a free idempotent system whose base $X$ is equipped with the structure of a Hausdorff topological space. According to 4.6, the topology on $X \subset U$ induces a Hausdorff topology on $U$. We refer to it as the *primary topology* on $U$. The following assertion immediately results from the definition of this topology.

**Proposition 4.8.** *For any open subset $V \subset U$ and any $f \in F$, the set $f(V, \dots, V)$ is also open.*

Introduce a new topology on $U$. For a base of neighborhoods we take the family of $F$-subsets $\widetilde{U'} = U(U') \subset U$ generated by open subsets $U' \subset U$ with respect to the primary topology. We refer to the topology thus obtained as the *secondary topology* on $U$.

It follows from Proposition 4.8 that any subset open in the secondary topology is open in the primary topology as well; however, the converse assertion fails. Thus, the secondary topology on $U$ is weaker than the primary topology.

As was shown above, the primary topology on $U$ induced by a Hausdorff topology on the base $X$ is also Hausdorff.

**Theorem 4.4.** *If the topology on the base $X \subset U$ is Hausdorff, then the induced secondary topology on $U$ is also Hausdorff.*

**Proof.** Let $x$ and $y$, $x, y \in U$, be arbitrary elements. We claim that, if $x \neq y$, then there exist neighborhoods $V_x$ and $V_y$ of the elements $x, y \in U$ with respect to the primary topology on $U$ such that the $F$-subsets $U_x = U(V_x)$ and $U_y = U(V_y)$ in $U$ generated by $V_x$ and $V_y$, respectively, are disjoint.

Denote by $Z$ a finite subset formed by all elements $z \in X$ subordinated to $x$ or $y$ and by $U'$ the $F$-subset of $U$ generated by $Z$, i.e., $U' = U[Z]$.

Let $\{V_u \mid u \in U'\}$ be an arbitrary system of neighborhoods (compatible with $U'$) in the primary topology on $U$ and let $U_u = U(V_u)$, $u \in U'$, be the $F$-subsets in $U$ generated by the corresponding neighborhoods. Note that, since the neighborhood system is compatible, each subset $V_u$ consists of elements of the same height. Therefore, this subset forms a base in $U_u$. We claim that $U_u \cap U_v = \varnothing$ for any $u, v \in U'$, $u \neq v$. In particular, since $x, y \in U'$, this will imply the assertion of the theorem.

Let us carry out the proof by induction on the height of the elements $u$ and $v$. We assume first that $h(u) = h(v) = 1$, i.e., $u, v \in Z$, and thus $V_u \subset X$ and $V_v \subset X$. Then it follows from the condition $V_u \cap V_v = \varnothing$ that $U_u \cap U_v = \varnothing$.

Let the assertion be already proved for the elements $u$ and $v$ of height less than $n$, and let $\max\{h(u), h(v)\} = n$, where $n > 1$. Suppose that the set $W = U_u \cap U_v$ is nonempty. Let $w \in W$ be an element of the minimal height. Then $w$ belongs to one of the neighborhoods, to $V_u$ or $V_v$. Indeed, otherwise the element $w$ would be simultaneously representable in the forms

$$w = f_1(u_1, \dots, u_m), \quad \text{where} \quad u_i \in U_u, \quad h(u_i) < h(w)$$

and

$$w = f_2(v_1, \dots, v_n), \quad \text{where} \quad v_i \in U_v, \quad h(v_i) < h(w).$$

By the uniqueness of decompositions, this would imply that $f_1 = f_2$, $m = n$, and $u_i = v_i$ for any $i$; thus, $u_i \in W$. Since $h(u_i) < h(w)$, this would contradict the condition that the element $w$ is minimal.

To be definite, let $w \in V_u$, and hence $w \notin V_v$. Then $h(w) = h(u)$ and $h(w) > h(v)$, and therefore $h(u) = n$ and $h(v) < n$. Since $h(u) > 1$, it follows that the element $u$ can be represented in the form

$$u = f(u_1, \dots, u_m), \quad \text{where} \quad u_i \in U', \quad h(u_i) < n.$$

Since $V_u = f(V_{u_1}, \dots, V_{u_m})$ by virtue of the compatibility condition for the neighborhoods, it follows that $w = f(u'_1, \dots, u'_m)$, where $u'_i \in V_{u_i}$, $i = 1, \dots, m$. On the other hand, since $w \in U_v$ and $w \notin V_v$, the element $w$ can be represented in the form $w = f'(v'_1, \dots, v'_n)$, where $v'_i \in U_v$. By the uniqueness condition for decompositions, this implies that $f = f'$, $m = n$, and $u'_i = v'_i$, and therefore $V_{u_i} \cap U_v \neq \varnothing$ for any $i$.

Let us prove that this is impossible. Indeed, it follows from the idempotent condition that at least two elements of the form $u_i$ in the decomposition $u = f(u_1, \dots, u_m)$ are distinct, and therefore $u_i \neq v$ for at least one index $i$. Then, since $h(u_i) < n$ and $h(v) < n$, it follows from the induction assumption that $U_{u_i} \cap U_v = \varnothing$.

**Corollary.** *The topology induced by the secondary topology on $U$ on the family $L = L(U, F)$ of the finite-rank $F$-subsets of a free idempotent $A$-system is Hausdorff. In this topology, the family $L^0 \subset L$ of the planar subsets is dense in $L$.*

**Remark.** The assertion of the theorem fails if the $A$-system in question is not idempotent. Indeed, for any $A$-system of this kind, there exists elements $x$ and $y$, $x, y \in U$, $x \neq y$, and operations $f \in F$ such that $x = f(y, \dots, y)$. In this case, any $F$-subset containing $y$ contains the element $x$ as well.

## 5. METRIC STRUCTURES ON $A$-SYSTEMS AND ON THE SETS OF THEIR SUBSYSTEMS

### 5.1. *Metric $A$-Systems*

Similarly to topological $A$-systems, there are two types of metric $A$-systems, namely, the $A$-systems $A = (U, F)$ with metric (Archimedean or non-Archimedean) defined on the support of $U$ only, and the $A$-systems with metric given not only on the support $U$ but also on the fundamental set $F$.

**Definition.** An $A$-system $A = (U, F)$ is said to be an *AM-system* if the support $U$ of this system is equipped with the structure of a metric space such that the mapping

$$(u_1, \dots, u_n) \in U^{\times n} \to f(u_1, \dots, u_n) \in U$$

is continuous for any $n$ and any $f \in F$.

**Definition.** An $A$-system $A = (U, F)$ is said to be an *AMF-system* if the sets $U$ and $F$ are equipped with the structures of metric spaces such that the mapping

$$(f, u_1, \dots, u_n) \in F_n \times U^{\times n} \to f(u_1, \dots, u_n) \in U$$

is continuous for any $n$.

If the topology on $F$ is discrete, then these definitions are equivalent.

If an $A$-system $A = (U, F)$ is an $N$-system, then every $F$-subset $U' \subset U$ admits a unique base. Using this fact, we can define, in terms of the metric (Archimedean or non-Archimedean) on the support $U$ of the $A$-system, a metric (Archimedean or non-Archimedean, respectively) on the family $L_n = L_n(U, F)$ of all $F$-subsets $U' \subset U$ of any given finite rank $n$. This metric on $L_n$ can be introduced in several different ways.

If $\rho$ is an Archimedean metric on $U$, then we can set, for instance,

$$\rho(U_1, U_2) = \min_\sigma \sqrt{\rho^2(x_1, y_{\sigma(1)}) + \dots + \rho^2(x_n, y_{\sigma(n)})}$$

for any $F$-subsets $U_1$ and $U_2$ with bases $X = \{x_1, \dots, x_n\}$ and $Y = \{y_1, \dots, y_n\}$, respectively, where the minimum is taken over all permutations $\sigma$ of the indices $1, \dots, n$.

Another way to define a metric on $L_n$ is

$$\rho(U_1, U_2) = \min_\sigma \Big( \max(d(x_1, y_{\sigma(1)}), \dots, d(x_n, y_{\sigma(n)})) \Big). \tag{11}$$

Let us show that, if $\rho$ is an Archimedean or non-Archimedean metric on $U$, then formula (11) defines an Archimedean or non-Archimedean metric $L_n$, respectively, i.e., for any $F$-subsets $U_1$, $U_2$, and $U_3$ of rank $n$, in the Archimedean case, the triangle inequality $\rho(U_1, U_3) \leqslant \rho(U_1, U_2) + \rho(U_2, U_3)$ holds, and in the non-Archimedean case, we have the stronger condition $\rho(U_1, U_3) \leqslant \max(\rho(U_1, U_3), \rho(U_1, U_3))$.

Indeed, let $X = \{x_1, \dots, x_n\}$, $Y = \{y_1, \dots, y_n\}$, and $Z = \{z_1, \dots, z_n\}$ be bases on $U_1$, $U_2$, and $U_3$, respectively. It follows from the definition of the metric $\rho$ on $L_n$ that there are permutations $\sigma_1$ and $\sigma_2$ such that

$$\rho(U_1, U_2) = \max(\rho(x_1, y_{\sigma_1(1)}), \dots, \rho(x_n, y_{\sigma_1(n)})),$$
$$\rho(U_2, U_3) = \max(\rho(y_{\sigma_1(1)}, z_{\sigma_1\sigma_2(1)}), \dots, \rho(y_{\sigma_1(n)}, z_{\sigma_1\sigma_2(n)})).$$

Hence,

$$\rho(U_1, U_2) + \rho(U_2, U_3) \geqslant \max(\rho(x_i, y_{\sigma_1(i)}) + \rho(y_{\sigma_1(i)}, z_{\sigma_1\sigma_2(i)}))$$

and

$$\max(\rho(U_1, U_2), \rho(U_2, U_3)) \geqslant \max(\rho(x_i, y_{\sigma_1(i)}), \rho(y_{\sigma_1(i)}, z_{\sigma_1\sigma_2(i)})).$$

If the metric on $U$ is Archimedean, then it follows from the first inequality that

$$\rho(U_1, U_2) + \rho(U_2, U_3) \geqslant \rho(x_i, z_{\sigma_1\sigma_2(i)})$$

for any $i = 1, \dots, n$, and therefore $\rho(U_1, U_2) + \rho(U_2, U_3) \geqslant \rho(U_1, U_3)$. If the metric on $U$ is non-Archimedean, then it follows from the other inequality that

$$\max(\rho(U_1, U_2), \rho(U_2, U_3)) \geqslant \rho(x_i, z_{\sigma_1\sigma_2(i)})$$

for any $i$, $i = 1, \dots, n$, and therefore $\max(\rho(U_1, U_2), \rho(U_2, U_3)) \geqslant \rho(U_1, U_3)$.

### 5.2. *Free AM-Systems with Archimedean and Non-Archimedean Metric*

Let $A = (U, F)$ be a free $A$-system, and let the base $X \subset U$ of this system be equipped with an Archimedean or non-Archimedean metric $\rho(x, y)$. Let us construct an extension of this metric to the entire set $U$.

We first define the metric $\rho$ by induction on the height of elements on all $S$-subsets $U_S \subset U$, i.e., on the subsets of elements with fixed decomposition schemes $S$. On the $S$-subset of elements of unit height, i.e., on the subset $X$, we have the initial metric $\rho$ (which is already Archimedean or non-Archimedean). Suppose that the metric $\rho$ is already defined on all $S$-subsets with elements of height less than $k$, and let $U_S$ be an arbitrary $S$-subset with elements of height $k$.

According to the definition of an $S$-subset, one can find an $f \in F$ and an $S$-subset $U_{S_1}, \dots, U_{S_n}$ with elements of height less than $k$, where $n$ is the arity of $f$, such that the $S$-subset $U_S$ consists of the elements of the form

$$u = f(u_1, \dots, u_n), \qquad h(u_i) < h(u), \quad \text{where} \quad u_i \in U_{S_i}, \qquad i = 1, \dots, n,$$

i.e.,

$$U_S = U_{S_1} \times \dots \times U_{S_n}.$$

By the induction assumption, the metric on the $S$-subsets $U_{S_i}$ is already defined. If this metric is Archimedean, then we define $\rho(u, v)$ for any elements $u = f(u_1, \dots, u_n)$ and $v = f(v_1, \dots, v_n)$ in $U_S$ by the formula

$$\rho(u, v) = \sqrt{\rho^2(u_1, v_1) + \dots + \rho^2(u_n, v_n)}. \tag{12}$$

Obviously, the function $\rho$ on $U_S \times U_S$ satisfies all axioms of Archimedean metric.

**Remark.** Certainly, the way of defining an Archimedean metric on $U_S$ is not unique. For instance, a metric can also be defined by the formula

$$\rho(u, v) = a_1\, \rho(u_1, v_1) + \dots + a_n\, \rho(u_n, v_n),$$

where $a_i$ are arbitrarily chosen positive numbers.

For the case in which the metric $\rho$ on the $S$-subsets $U_{S_i}$ is non-Archimedean, let us define $\rho(u, v)$ for any elements $u = f(u_1, \dots, u_n)$ and $v = f(v_1, \dots, v_n)$ in $U_S$ by the formula

$$\rho(u, v) = \max(\rho(u_1, v_1), \dots, \rho(u_n, v_n)). \tag{13}$$

Obviously, the metric thus defined on $U_S$ is non-Archimedean.

Let us extend the metric $\rho$ defined on the $S$-subsets in $U$ to the entire set $U$. For any decomposition scheme $S$, we choose an arbitrary element $y_S \in U_S$. If the metric on the $S$-subsets $U_S$ is Archimedean, then, for any $u \in U_{S_1}$ and $v \in U_{S_2}$ with $S_1 \neq S_2$, we set

$$\rho(u, v) = \rho(u, y_{S_1}) + \rho(v, y_{S_2}) + a,$$

where $a$ is an arbitrarily chosen positive number.

One can readily see that the metric $\rho$ thus defined on $U$ is Archimedean. Indeed, it suffices to verify the triangle inequality for any elements $u \in U_{S_1}$, $v \in U_{S_2}$, and $w \in U_{S_3}$ in the following two cases:

(1) $S_1 = S_2 \neq S_3$,

(2) $S_1$, $S_2$, and $S_3$ are pairwise distinct.

In the first case, we have

$$\rho(u, w) = \rho(u, y_{S_1}) + \rho(w, y_{S_3}) + a,$$
$$\rho(v, w) = \rho(v, y_{S_2}) + \rho(w, y_{S_3}) + a,$$
$$\rho(u, v) \leqslant \rho(u, y_{S_1}) + \rho(v, y_{S_2}) + a.$$

In the other case, the last inequality is replaced by the corresponding equality. In both cases, this implies the triangle inequality.

If the metric on the $S$-subsets $U_S$ is non-Archimedean, for any $u \in U_{S_1}$ and $v \in U_{S_2}$, where $S_1 \neq S_2$, we set

$$\rho(u, v) = \max(\rho(u, y_{S_1}),\ \rho(v, y_{S_2})\, a),$$

where $a$ is an arbitrarily fixed positive number.

Similarly to the Archimedean case, one can readily see that the metric $\rho$ thus defined on $U$ is non-Archimedean.

The $AM$-systems thus defined are said to be *free $AM$-systems* (Archimedean or non-Archimedean, respectively).

Since the metric on the base $X$ of a free $AM$-system induces the topology on $X$, to any free $AM$-system one can assign a free $AT$-system. The following assertion immediately results from the definitions of free $AT$- and $AM$-systems.

**Proposition 5.1.** *The topology on the support $U$ of a free $AM$-system $A = (U, F)$ induced by an Archimedean or non-Archimedean metric on $U$ coincides with the topology on $U$ induced by the corresponding $AT$-system.*

An extension of an Archimedean or non-Archimedean metric with base $X \subset U$ to the support $U$ can be defined in a similar way for each free idempotent or free commutative $A$-system $A = (U, F)$. The definition can be extended to all free idempotent $A$-systems without modifications. For a free commutative system, relations (12) and (13) in the definition of the distance must be replaced by the relations

$$\rho(u, v) = \min_{\sigma} \sqrt{\rho^2(u_1, v_{\sigma(1)}) + \cdots + \rho^2(u_n, v_{\sigma(n)})}$$

and

$$\rho(u, v) = \min_{\sigma}(\max(\rho(u_1, v_1), \ldots, \rho(u_n, v_n))),$$

respectively, where the minimum is taken over all permutations $\sigma$ of the indices $1, \ldots, n$.

Thus, two new families of $AM$-systems with Archimedean and non-Archimedean metric arise, namely, free idempotent and free commutative $AM$-systems.

### 5.3. *Secondary Non-Archimedean Metric on Free Idempotent A-Systems*

Let $A[X] = (U, F)$ be a free idempotent $A$-system, and let its base $X \subset U$ be equipped with a non-Archimedean metric $d$. Let us construct an extension of this metric to the support $U$ of the $A$-system; this extension differs from that in Subsection 5.2.

Let us define a metric $d$ on $S$-subsets of $U$ in the same way as in Subsection 5.2, i.e., by formula (13), using the induction on the height of elements.

Let us extend the metric $d$ to the entire set $U$, i.e., let us define $d(u, v)$ for any elements $u, v \in U$ belonging to different $S$-subsets. To be definite, let $h(u) \leqslant h(v)$. Then $h(v) > 1$, and therefore the element $v$ can uniquely be represented in the form

$$v = f(v_1, \ldots, v_k), \quad \text{where} \quad h(v_i) < h(v), \qquad i = 1, \ldots, k.$$

If $h(u) < h(v)$, then we set

$$d(u, v) = \max(d(u, v_1), \ldots, d(u, v_k)). \tag{14}$$

If $h(u) = h(v)$, then we represent $u$ in the form

$$u = f_1(u_1, \ldots, u_l), \quad \text{where} \quad h(u_i) < h(u), \qquad i = 1, \ldots, l,$$

and set

$$d(u, v) = \max_{i,j}(d(u_i, v_j)), \tag{15}$$

where the maximum is taken over all $i$ and $j$, $i = 1, \ldots, l$ and $j = 1, \ldots, k$.

**Proposition 5.2.** *The function $d(u,v)$ on $U$ thus defined satisfies the axioms of non-Archimedean metric.*

**Proof.** It suffices to show that 1) $d(u,v) \neq 0$ if $u$ and $v$ belong to distinct $S$-subsets, and 2) $d(u,w) \leqslant \max(d(u,v), d(v,w))$ if $u$, $v$, and $w$ do not belong simultaneously to the same $S$-subset.

Let us carry out the proof by induction on the height of elements. The assertion is trivial for the elements of unit height because these elements belong to the same $S$-subset. Suppose that the assertion is already proved for the elements of height less than $n$.

Suppose that $d(u,v) = 0$ for some elements $u, v$ belonging to different $S$-subsets and assume that $h(u) \leqslant h(v) = n$.

If $h(u) < h(v)$, then $d(u,v)$ is given by relation (14), where $v_i$ are elements of height less than $n$ that enter the representation of the element $v$, $v = f(v_1, \dots, v_k)$. Hence, $d(u, v_i) = 0$ for any $i$. Then it follows from the induction assumption that $v_i = u$ for any index $i$. Thus, by the idempotence property, we have $v = f(u, \dots, u) = u$, which is wrong.

If $h(u) = h(v)$, then $d(u,v)$ is given by relation (15), where any $u_i$ is an element of height less than $n$ entering the representation of the element $u$, $u = f(u_1, \dots, u_l)$. Hence, $d(u_i, v_j) = 0$ for any $i$ and $j$. In this case, it follows from the induction assumption that the elements $u_i$ and $v_j$ are equal. By the idempotence property, we have $v = u$, which is wrong. Thus, property 1 is proved.

Now let us prove the inequality $d(u,w) \leqslant \max(d(u,v), d(v,w))$ for any elements $u$, $v$, and $w$ that satisfy the condition $\max(h(u), h(v), h(w)) = n$ and do not belong simultaneously to any $S$-subset. The following cases are possible:

(1) $h(u) = h(v) = h(w)$, where either $S(u) \neq S(v) = S(w)$ or $S(u)$, $S(v)$, $S(w)$ are pairwise distinct;
(2) $h(u) < h(v) = h(w)$, where either $S(v) \neq S(w)$ or $S(v) = S(w)$;
(3) $h(u) = h(v) < h(w)$, where either $S(u) \neq S(v)$ or $S(u) = S(v)$;
(4) $h(u) < h(v) < h(w)$.

One can immediately see that condition 2 (for each permutation of the elements $u$, $v$, and $w$) holds in each of these cases.

We refer to the non-Archimedean metric on $U$ thus introduced as the *secondary non-Archimedean metric* associated with the given non-Archimedean metric on the base $X \subset U$.

**Example.** Let $G[X]$ be the free idempotent groupoid with the base $X = \{x, y, z\}$, and let a non-Archimedean metric $d$ be given on $X$. Let us evaluate the distance between the elements $z_1 = xz$ and $z_2 = (xy)z$ in the secondary metric on $G[X]$.

Since $h(xz) < h((xy)z)$, it follows that

$$d(xz, (xy)z) = \max(d(xz, xy), d(xz, z)).$$

Further, since the decomposition schemes of the elements $xz$ and $zy$ coincide and since $h(xz) > h(z)$, it follows that

$$d(xz, xy) = \max(d(x,x), d(z,y)) = d(z,y),$$
$$d(xz, z) = \max(d(x,z), d(z,z)) = d(x,z).$$

Thus,

$$d(xz, (xy)z) = \max(d(z,y), d(x,z)).$$

**Remark.** In the definition of the secondary metric, the assumption that the $A$-system in question is idempotent turns out to be substantial. Indeed, otherwise there exist elements of the form $v = f(u, \dots, u)$, where $h(u) < h(v)$. By (14), for these elements, we have $d(u,v) = 0$, which contradicts the definition of the metric $d$.

Let us compare the secondary metric on $U$ with the metric introduced in Subsection 5.2 (which we call *primary metric*). It follows from the definition of these metrics that they coincide on any

$S$-subset $U_S \subset U$ with a fixed decomposition scheme $S$. At the same time, the subset $U_S$ is open in the topology on $U$ induced by the primary metric but is not open with respect to the topology on $U$ induced by the secondary metric. Thus, the secondary metric induces a topology on $U$ which is weaker than the primary one.

Let $A[X] = (U, F)$ be a free idempotent system with secondary metric generated by a non-Archimedean metric $d$ on the base $X$ of the system. Denote by $\Omega = \Omega_r(u)$ the ball of radius $r$ in $U$ centered at the point $u \in U$, i.e.,

$$\Omega_r(u) = \{v \in U \mid d(u,v) \leqslant r\}.$$

Note that, in a non-Archimedean metric, any point of the ball is a center of this ball.

**Theorem 5.1.** *Let $k$ be the least height of the elements belonging to the ball $\Omega_r(u)$, and let $V \subset \Omega_r(u)$ be the subset of all elements of height $k$ in this ball. Then*

$$\Omega_r(u) = U(V),$$

*where $\widetilde{V} = U(V)$ is the $F$-subset of $U$ generated by $V \subset U$.*

**Proof.** Since for the center of the ball $\Omega$ one can take each point $u$ of this ball, we can assume that $u \in V$. Then

$$h(u) = k \quad \text{and} \quad V = \{v \in U \mid h(v) = k,\ d(u,v) \leqslant r\}.$$

Let us first prove that $\widetilde{V} \subset \Omega$. Denote by $V^{(n)}$ the subset of elements of height $n$ in $\widetilde{V} = U(V)$ with respect to $V$. By definition, $V^{(1)} = V$, and therefore $V^{(1)} \subset \Omega$. Assume that $V^{(m)} \subset \Omega$ for any $m < n$, and let $v \in V^{(n)}$. Represent $v$ in the form $v = f(v_1, \dots, v_l)$, where $h(v_i) < h(v)$, $i = 1, \dots, l$. By the induction assumption, we have $v_i \in \Omega$, and therefore $d(u, v_i) \leqslant r$, $i = 1, \dots, l$. Since $d(u,v) = \max_i(d(u,v_i))$ by the definition of the metric $d$, it follows that $d(u,v) \leqslant r$, and therefore $v \in \Omega$.

Conversely, let $v \in \Omega$. Then $h(v) \geqslant k$. If $h(v) = k$, then $v \in V$, and therefore $v \in U(V)$. Assume that all elements $v \in \Omega$ of height less than $n$ (where $n > k$) belong to $U(V)$, and let $h(v) = n$. Let us represent $v$ in the form $v = f(v_1, \dots, v_l)$, where $h(v_i) < h(v)$, $i = 1, \dots, l$. Since $d(u,v) = \max_i(d(u,v_i))$, it follows from $d(u,v) \leqslant r$ that $d(u,v_i) \leqslant r$, and therefore $v_i \in \Omega$, $i = 1, \dots, l$. Thus, $v_i \in U(V)$, $i = 1, \dots, l$, by the induction assumption. Then $v \in U(V)$.

## REFERENCES


1. Kurosh, A. G., *Lectures on General Algebra*, M.: Gosudarstv. Izdat. Fiz.-Mat. Lit., 1962; English transl. in: *Lectures in General Algebra*, Oxford: Pergamon, 1965.
2. Cohn, P. M., *Universal Algebra*, Dordrecht: D. Reidel, 1981.
3. Borůvka, O., *Grundlagen der Gruppoid- und Gruppenteorie*, Berlin: VEB Deutscher Verlag der Wissenschaften, 1960; English transl. in: Boruvka, O., *Foundations of the Theory of Groupoids and Groups*, New York: Halsted, 1976.
4. Birkhoff, G., *Lattice Theory*, 3rd ed., AMS Colloquium Publications, vol. XXV, 1996.
5. Stern, M., *Semimodular Lattices. Theory and Applications*, Cambridge: Cambridge University Press, 1999.