A topos-theoretic approach to branching space-time

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1 Introduction

Isham and Döring [1] put forward a fundamental conception of how any physical theory should be constructed. They proposed to look for topos representation of the formal language used in the theory. The known and chronologically preceding example of such an approach is the work [2] by Fotini Markopoulou. She considered from a novel view-point the properties of the causal set $\mathcal{C}$ which is the discrete analog of space-time from general relativity. The interest to causal sets is stipulated by hypothetical discreteness of space-time at sub-Planck scale. The causal set $\mathcal{C}$ is also the central notion of the present work. It is a partly ordered set of all events. Some ordered event pairs $\langle e, e' \rangle$ in $\mathcal{C}$ are causally related: $e \rightsquigarrow e'$. That means that e' is a consequence of e or, equivalently, e is a cause of e'. The causal relation is reflexive (for any e in $\mathcal{C}$ one has $e \rightsquigarrow e$), transitive (from $e \rightsquigarrow e'$ and $e' \rightsquigarrow e''$ follows $e \rightsquigarrow e''$) and antisymmetric (if $e \rightsquigarrow e'$ and $e' \rightsquigarrow e$, then $e = e'$). The last condition guarantees the absence of closed causal loops. We also assume the absence of the last event e_{fin} which is the consequence of any other event in $\mathcal{C}$. This is important for the subject of the present work being non-trivial.

The partial order in $\mathcal{C}$ let one consider it as a category with the events as objects and causal relations between them as arrows (morphisms). So the set $Mor_{\mathcal{C}}(e, e')$ of morphisms from e to e' consists of at most of one element just in the case when e is the cause of e'. Using the notions of category theory, Markopoulou introduced the (covariant) functor $Past$ from the category $\mathcal{C}$ to the category of sets Set. To any event e from $\mathcal{C}$ this functor associates the set $Past_e = \{e' \in \mathcal{C} : e' \rightsquigarrow e\}$ (the set of all causes with respect to e), and to any causal arrow $e_1 \rightsquigarrow e_2$ – the map $Past_{e_1 e_2} : Past_{e_1} \to Past_{e_2}$ which is the inclusion of sets.

The functor $Past$ from [2] has simple physical meaning: $Past_e$ is the memory content for an observer localized in e. Note that the complete causal structure of space-time is reflected in $Past$. The approach of the present work to $\mathcal{C}$ is also realized as a view-point of localized observers. But some other properties of the causal set are in focus. We are going to address the branching properties of space-time. The very notion of "branching" will get the strict sense. The motivation beyond the approach is to elaborate a logical framework for considering multivariant future by a local observer. So the subtitle of the present work (in the style of [2] subtitle) can be the phrase "What should the branching Universe be thought of from the inside?".

We are going to build the topos representation of Nuel Belnap's branching space-time theory [3]. The aim of the branching space-time conception is the reconciliation and unification of indeterminism and relativism. In the following section the central notion of the branching space-time theory, the so-called Belnap "world" (Belnaps calls it "history"), is introduced along with basic notions of categories. Because the ideas of topos mathematics has not yet become widely known, the present work is aimed to be in its considerable part an introduction to basic notions of topoi. In the third section we give the main elements of topos approach in application to the model of branching space-time and show the origin of its natural non-classical logic. The application of universal topos construction of local semantic values to propositions of the natural logic is made in the fourth section.

2 Basic notions

First we provide and discuss the required formal and methodological tools. Let us digress for a while on the main notions of category theory. It is known [4], that any category C is specified by its objects and morphisms (arrows) between them. For some arrows the following composition law is defined. Let $Mor_C(c, c')$ be the set of arrows (morphisms) beginning and ending at the objects c and c', respectively. There is a rule by which to any ordered pair of morphisms from the direct product $Mor_C(c_1, c_2) \times Mor_C(c_2, c_3)$ an element (morphism) from $Mor_C(c_1, c_3)$ is assigned, their composit. The composition law is associative and in every $Mor_C(c, c)$ there is an identity morphism 1_c which acts as a left unit in compositions with elements from $Mor_C(c, c')$ and as a right unit in compositions with elements from $Mor_C(c', c)$. The most important category is Set with sets as objects and maps between sets as morphisms.

Covariant functor $\mathcal{F}$ from category C_1 to category C_2 is a rule of "projecting" the category structure of C_1 onto C_2. To every object c from C_1 an object $\mathcal{F}_c$ from C_2 is assigned. Similarly, the functor assignes to every morphism f from $Mor_{C_1}(c, c')$ some morphism $\mathcal{F}_f$ from $Mor_{C_2}(\mathcal{F}_c, \mathcal{F}_{c'})$. The identity morphisms as well as composits are respected by the functor. The specific of contravariant functor is in inversion of morphism directions: to f from $Mor_{C_1}(c, c')$ there assigned the morphism $\mathcal{F}_f$ from $Mor_{C_2}(\mathcal{F}_{c'}, \mathcal{F}_c)$.

The functors from a category C_1 to a category C_2 can be considered as objects of a new category $C_2^{C_1}$. Morphisms between such functors are called natural transformations. The examples of natural transformations will appear later in the text.

We need not the strict definition of topos as a special type of category [5]. In some sense all topos are like the classical topos Set – the category of sets. It is important for us that if C_2 is a topos, then $C_2^{C_1}$ is a topos also. Below the topos $Set^{\mathcal{C}}$ is used.

Considering $\mathcal{C}$ as a branching space-time, it is convenient to introduce subsets from $\mathcal{C}$ without branching and, hence, without indeterminism at all. The richer is the collection of such branchless subsets, more branching is the causal set $\mathcal{C}$. These subsets will be called *Belnap worlds* and, following [3], are defined as *maximal upward (i.e. towards future) directed subsets of* $\mathcal{C}$.

The demand of being directed is natural and motivated by evidently indispensable property of any branchless world – for any two events e_1 и e_2 in a world w, there should be their common consequence e in w: $e_1 \rightsquigarrow e$ and $e_2 \rightsquigarrow e$. The maximality does not allow a world to be a proper sub-world in a wider world. Now the condition of absence of the last event e_{fin} ("Big flap") in $\mathcal{C}$ is clear. In other case $\mathcal{C}$ is the single maximal directed set and the subject of the present work becomes trivial.

The standard application of Kuratovski-Zorn's lemma proves any event pertaining to some Belnap world: given any growing chain of directed sets which contain the event, one can note that their union is an upper bound of the chain. Thus a maximal directed subset in $\mathcal{C}$ exists and contains the event. There may happen no Belnap world for a pair of events e_1 and e_2 to live in. Such events are called *incompatible.* The alternative outcomes of a quantum measurement is an example of incompatible events. One should not mix the notion of incompatible events with causally unrelated events.

3 Main object and subobjects

We are going to deal with some contravariant functors, called presheaves, from $\mathcal{C}$ to Set, i.e. the objects of the topos $Set^{\mathcal{C}^{op}}$, where the category $\mathcal{C}^{op}$ can be formed, given by $\mathcal{C}$, by reversing the direction of all causal arrows. In this point our approach differs with that of [2], where covariant functors are used.

Let W be the set of all Belnap worlds in $\mathcal{C}$. There is a simple but important

Theorem 3.1. *Assigning to an event e the set of Belnap worlds $\mathbf{Loc}_e = \{w \in W : e \in w\}$, one defines the functor* **Loc** *from $\mathcal{C}^{op}$ to Set.*

Proof: The sets $\mathbf{Loc}_e$ define the function of objects of the functor. We have to clarify the nature of function of causal arrows $e_1 \rightsquigarrow e_2$ from $\mathcal{C}$. This should be a map

$$\mathbf{Loc}_{e_1e_2} : \mathbf{Loc}_{e_2} \to \mathbf{Loc}_{e_1}, \tag{1}$$

The map is set inclusion. To prove the fact one should involve the maximality of Belnap worlds, which lead to the closeness of any world with respect to causes [3]: if e is a cause of $e' \in w$, i.e. $e \rightsquigarrow e'$, there is in w a common consequence of e and e'', where e'' is any event in w (due to transitivity of causal relation this common cause can be chosen among common causes of e' and e'' which exist because of the directed nature of the world). Hence, the world w can be extended by inclusion of e. But w is maximal and can not be extended – the event e has been already included in w. It follows that every world containing e_2, contains e_1 as well and, so, $\mathbf{Loc}_{e_1e_2}$ (1) is inclusion.

□

The functor **Loc** is in some sense analogous to the functor $Past$ from [2] and is of importance also. $\mathbf{Loc}_e$ is the set of worlds an observer localized in e considers as "her worlds".

It is worth to introduce the presheaf **Glob** (the counterpart of the functor $World$ from [2]):

$$\mathbf{Glob}_e = W, \tag{2}$$

and for $e_1 \rightsquigarrow e_2$

$$\mathbf{Glob}_{e_1e_2} = id_W : \mathbf{Glob}_{e_2} \to \mathbf{Glob}_{e_1}, \tag{3}$$

There can be introduced morphisms between functors from $\mathcal{C}^{op}$ to Set, considered as objects of the category $Set^{\mathcal{C}^{op}}$. The morphisms are called natural transformations [4]. In particular, the natural transformation

$$[\iota_{\mathbf{Loc}}] : \mathbf{Loc} \to \mathbf{Glob} \tag{4}$$

is the set $\{[\iota_{\mathbf{Loc}}]_e : e \in \mathcal{C}\}$ of maps $[\iota_{\mathbf{Loc}}]_e : \mathbf{Loc}_e \to \mathbf{Glob}_e$,which make the following diagrams commutative:

$$\begin{array}{ccc} \mathbf{Loc}_{e_2} & \xrightarrow{[\iota_{\mathbf{Loc}}]_{e_2}} & \mathbf{Glob}_{e_2} \\ \mathbf{Loc}_{e_1e_2}\downarrow & & \downarrow \mathbf{Glob}_{e_1e_2 = id_W} \\ \mathbf{Loc}_{e_1} & \xrightarrow{[\iota_{\mathbf{Loc}}]_{e_1}} & \mathbf{Glob}_{e_1} \end{array} \tag{5}$$

for any causal arrow $e_1 \rightsquigarrow e_2$, i.e. $id_W \circ [\iota_{\mathbf{Loc}}]_{e_2} = [\iota_{\mathbf{Loc}}]_{e_1} \circ \mathbf{Loc}_{e_1e_2}$. In this simple case the components $[\iota_{\mathbf{Loc}}]_e$, of the natural transformation (4) are set inclusions. This let the functor **Loc** be considered as a sub-functor of **Glob**. If one calls **Glob** the object in $Set^{\mathcal{C}^{op}}$, then **Loc** is the *subobject*.

The set of subobjects of **Glob** is important in the topos approach to the branching space-time. The set of subobjects in any topos is known to be endowed with the structure of Heyting algebra [5]. It is a model of a natural language of the considered system (the branching space-time $\mathcal{C}$) [1]. The logic of the language is not classical one based on Boolean algebra. Logical operations are realized as algebraic operations on subobjects of **Glob**. The conjunction $\mathbf{F} \wedge \mathbf{G}$, models the logicaoperation "and". For subobjects **F** and **G** one has

$$(\mathbf{F} \wedge \mathbf{G})_e =_{df} \mathbf{F}_e \cap \mathbf{G}_e. \tag{6}$$

Similarly, the disjunction $\mathbf{F} \vee \mathbf{G}$ models the operation "or" and is defined as follows:

$$(\mathbf{F} \vee \mathbf{G})_e =_{df} \mathbf{F}_e \cup \mathbf{G}_e. \tag{7}$$

For any causal arrow $e_1 \rightsquigarrow e_2$ the maps $\mathbf{F}_{e_1e_2}$ and $\mathbf{G}_{e_1e_2}$ are set inclusions (this follows from the corresponding (5)-type commutative diagrams, where the horizontal arrows are inclusions). Therefor, the maps

$$(\mathbf{F} \wedge \mathbf{G})_{e_1e_2} : (\mathbf{F} \wedge \mathbf{G})_{e_2} \to (\mathbf{F} \wedge \mathbf{G})_{e_1} \tag{8}$$

and

$$(\mathbf{F} \vee \mathbf{G})_{e_1e_2} : (\mathbf{F} \vee \mathbf{G})_{e_2} \to (\mathbf{F} \vee \mathbf{G})_{e_1}. \tag{9}$$

are set inclusions as well.

The binary operation $\mathbf{F} \Rightarrow \mathbf{G}$ models logical implication:

$$(\mathbf{F} \Rightarrow \mathbf{G})_e =_{df} \{w \in W : \forall e' \rightsquigarrow e \, (w \in \mathbf{F}_{e'}) \Rightarrow (w \in \mathbf{G}_{e'})\}. \tag{10}$$

Here $\Rightarrow$ in lhs means the binary operation on the functors **F** and **G**, the same symbol in rhs stands for the ordinary logical connection "if..., then...". It is easy to see that for any causal arrow $e_1 \rightsquigarrow e_2$ the set $(\mathbf{F} \Rightarrow \mathbf{G})_{e_2}$ is a subset in $(\mathbf{F} \Rightarrow \mathbf{G})_{e_1}$. Therefore, the functorial image of the causal arrow is the set inclusion:

$$(\mathbf{F} \Rightarrow \mathbf{G})_{e_1 e_2} : (\mathbf{F} \Rightarrow \mathbf{G})_{e_2} \rightarrow (\mathbf{F} \Rightarrow \mathbf{G})_{e_1}. \tag{11}$$

The implication operation let one define the unary operation of negation $\neg$ in the set of subfunctors of **Glob**. To this end one need the zero subfunctor $\emptyset$. It assigns the empty set to any event. Let

$$\neg\mathbf{F} =_{df} (\mathbf{F} \Rightarrow \emptyset). \tag{12}$$

From this definition and (10) follows

$$(\neg\mathbf{F})_e = \{w \in W : \forall e' \rightsquigarrow e \ (w \notin \mathbf{F}_{e'})\}. \tag{13}$$

The intersection $\mathbf{F}_e \cap (\neg\mathbf{F})_e$ is evidently the empty set. So

$$\mathbf{F} \wedge (\neg\mathbf{F}) = \emptyset. \tag{14}$$

In the general case

$$\mathbf{F} \vee (\neg\mathbf{F}) \neq \mathbf{Glob}. \tag{15}$$

If it were the place of exact equality in the last expression, one would be free to identify $\mathbf{F}$ and $\neg\neg\mathbf{F}$. But there is only a weaker statement that $\mathbf{F}$ is a subfunctor of $\neg\neg\mathbf{F}$:

$$\mathbf{F}_e \subseteq (\neg\neg\mathbf{F})_e. \tag{16}$$

Really, due to (13), we have

$$(\neg\neg\mathbf{F})_e = \{w \in W : \forall e' \rightsquigarrow e \, \exists e'' \rightsquigarrow e' \ (w \in \mathbf{F}_{e''})\}. \tag{17}$$

Because of $\mathbf{F}_e \subseteq \mathbf{F}_{e''}$, taking place as soon as $e'' \rightsquigarrow e$, the expression (16) follows. The reverse inclusion $(\neg\neg\mathbf{F})_e$ в $\mathbf{F}_e$ is generally incorrect, then $\mathbf{F}_e \subsetneq (\neg\neg\mathbf{F})_e$. according to (14), $(\neg\mathbf{F})_e \cap (\neg\neg\mathbf{F})_e = \varnothing$. Consequently, $(\neg\mathbf{F})_e \cup (\mathbf{F})_e$ is a proper subset of $(\neg\mathbf{F})_e \cup (\neg\neg\mathbf{F})_e$ which, in its turn, is a subset of W. Therefore, $(\neg\mathbf{F}) \vee \mathbf{F}$ is a proper subfunctor in **Glob**.

Expression (15) points the principe of excluded middle to be not fulfilled in the logic of subobjects in **Glob** (the presheaf **Glob** plays the role of identically true proposition). This is a generic property of topoi, other then *Set* [5]. The logic is intuitionistic.

As it is pointed in [2], some important properties of the causal set $\mathcal{C}$ are reflected in the structure of $\neg Past$, *viz* $\neg Past = \emptyset$ provided $\mathcal{C}$ is a lattice (here the functor $\emptyset$ along with $Past$, is the object of category $Set^{\mathcal{C}}$ and does not coincide with contravariant counterpart). This follows from the definition

$$(\neg Past)_e = \{e' \in \mathcal{C} : \forall e \rightsquigarrow e''(e' \notin Past_{e''})\}. \tag{18}$$

We see that if any pair of events has an upper bound, the set $(\neg Past)_e$ is empty. Reformulating this observation within our approach aimed at branching properties of $\mathcal{C}$, we arrive at the following

Proposition 3.1. *The subject of the present work (the branching of space-time) is non-trivial iff the functor $\neg Past$ from [2] is non-empty.*

. $\square$

As it has been pointed out, in our case **Loc** is an analog of $Past$. Consequently, one may expect the structure of **Loc** reflects some important properties of $\mathcal{C}$ and the set of worlds. Due to (13) we get:

$$(\neg \mathbf{Loc})_e = \{w \in W : \forall e' \rightsquigarrow e \;\; (e' \notin w)\}. \tag{19}$$

It is easy to verify that

$$(\neg \mathbf{Loc} = \emptyset) \Leftrightarrow (\forall e \in \mathcal{C} \; \forall w \in W \;\; Past_e \cap w \neq \varnothing). \tag{20}$$

So, we have

Proposition 3.2. *The emptiness of $\neg$**Loc** is equivalent to non-empty intersection of any Belnap world with the past of any event.*

. $\square$

Particularly, this is the case of $\mathcal{C}$ containing the initial event e_{in} ("Big Bang"), such that $e_{in} \rightsquigarrow e$ for any event e.

4 Subobject classifier

In $Set^{\mathcal{C}^{op}}$, as well as in each topos, there is a subobject classifier for any object. Intuitively, the subobject classifier delivers generalized truth values for a set of propositions. This can be illustrated by application of the general construction [5] in our setting.

Let us consider the set $Mor_{\mathcal{C}}(\cdot, e)$ of causal arrows ending in e. Special subsets called sieves on e are of importance. Any sieve $S \subseteq Mor_{\mathcal{C}}(\cdot, e)$ is closed in the following sense: if $(e' \rightsquigarrow e) \in S$, and there is a causal arrow $e'' \rightsquigarrow e'$, then $(e'' \rightsquigarrow e) \in S$. The maximal sieve on e is the very set $Mor_{\mathcal{C}}(\cdot, e)$. Its empty subset is the minimal sieve.

It is worth to introduce the set

$$\mathbf{\Omega}_e =_{df} \{S : S \text{ – sieve on } e\}. \tag{21}$$

For any causal arrow $e_1 \rightsquigarrow e_2$ there is a map

$$\mathbf{\Omega}_{e_1e_2} : \mathbf{\Omega}_{e_2} \to \mathbf{\Omega}_{e_1} \tag{22}$$

by the following rule: given the sieve S on e_2, one should consider

$$\mathbf{\Omega}_{e_1e_2}(S) =_{df} \{(e \rightsquigarrow e_1) \in Mor_{\mathcal{C}}(\cdot, e_1) : (e \rightsquigarrow e_2) \in S\} \tag{23}$$

as its image in $\mathbf{\Omega}_{e_1}$. Expressions (21) – (23) let us define the contravariant functor $\mathbf{\Omega}$ from $\mathcal{C}$ to Set.

For any subobject $\mathbf{F}$ from **Glob** with the corresponding inclusion

$$[\iota_{\mathbf{F}}] : \mathbf{F} \to \mathbf{Glob} \tag{24}$$

consider the following maps

$$[\chi_{\mathbf{F}}]_e : W \to \mathbf{\Omega}_e, \tag{25}$$

which let assign a sieve on e to any Belnap world:

$$[\chi_{\mathbf{F}}]_e(w) =_{df} \{(e' \rightsquigarrow e) \in Mor_{\mathcal{C}}(\cdot, e) : w \in \mathbf{F}_{e'}\}. \tag{26}$$

It is easy to see that this is really a sieve: if a causal pair $e' \rightsquigarrow e$ belongs to rhs of (26) and there is an arrow $e'' \rightsquigarrow e'$, then $w \in \mathbf{F}_{e''}$ because $\mathbf{F}_{e'} \subseteq \mathbf{F}_{e''}$ and, consequently, $e'' \rightsquigarrow e$ belongs to rhs of (26) as well.

The fact is of importance that the maps (25) (defined for all events) make the diagram

$$\begin{array}{ccc} W & \xrightarrow{[\chi_{\mathbf{F}}]_{e_2}} & \mathbf{\Omega}_{e_2} \\ \downarrow{\scriptstyle id_W} & & \downarrow{\scriptstyle \mathbf{\Omega}_{e_1e_2}} \\ W & \xrightarrow{[\chi_{\mathbf{F}}]_{e_1}} & \mathbf{\Omega}_{e_1}, \end{array} \tag{27}$$

commutative for any arrow $e_1 \rightsquigarrow e_2$. Therefore, $[\chi_{\mathbf{F}}]_e$ can be considered as components of the natural transformation

$$[\chi_{\mathbf{F}}] : \mathbf{Glob} \to \mathbf{\Omega}. \tag{28}$$

It follows from (26) that if $w \in \mathbf{F}_e$, then $[\chi_{\mathbf{F}}]_e(w) = Mor_{\mathcal{C}}(\cdot, e)$. Oppositely, if the last equality is fulfilled then from $e \rightsquigarrow e \in Mor_{\mathcal{C}}(\cdot, e)$ one has $w \in \mathbf{F}_e$. We see that $[\chi_{\mathbf{F}}]_e$ maps all the worlds from $\mathbf{F}_e$ and only these worlds to the maximal sieve on e. This makes the following diagram commutative

$$\begin{array}{ccc} \mathbf{F} & \xrightarrow{[\iota_{\mathbf{F}}]} & \mathbf{Glob} \\ \downarrow{\scriptstyle [!_{\mathbf{F}}]} & & \downarrow{\scriptstyle [\chi_{\mathbf{F}}]} \\ \mathbf{1} & \xrightarrow{\top} & \mathbf{\Omega} \end{array} \tag{29}$$

Here the functor $\mathbf{1}$ assigns the fixed one-element set $\{0\}$ to any event from $\mathcal{C}$ and the natural transformation components $[!_{\mathbf{F}}]_e$ are the only possible maps from $\mathbf{F}_e$ to $\{0\}$. The natural transformation $\top$ (truth) has the components $\top_e : \{0\} \to \mathbf{\Omega}_e$ such that $\top_e(0) = Mor_{\mathcal{C}}(\cdot, e)$ is the maximal sieve on e. $\mathbf{F}$ along with the natural transformations $[!_{\mathbf{F}}]$ and $[\iota_{\mathbf{F}}]$ are the pull-back of the diagram $\mathbf{Glob} \xrightarrow{[\chi_{\mathbf{F}}]} \mathbf{\Omega} \xleftarrow{\top} \mathbf{1}$ [5].

As has been said, the presheaf $\mathbf{Loc}$ is the most simple and important subobject of $\mathbf{Glob}$. To its inclusion in $\mathbf{Glob}$ (4), the following natural transformation corresponds

$$[\chi_{\mathbf{Loc}}] : \mathbf{Glob} \to \mathbf{\Omega}, \tag{30}$$

so that

$$[\chi_{\mathbf{Loc}}]_e(w) =_{df} \{(e' \rightsquigarrow e) \in Mor_{\mathcal{C}}(\cdot, e) : e' \in w\}. \tag{31}$$

The sieve on e defined by rhs of this expression, is the generalized truth value of an e-localized observer being living or having been lived in the world w. If the event e pertains to the world w, then this is the maximal sieve on e – the maximal truth value. Oppositely, if $Past_e \cap w = \varnothing$, the rhs of (31) is the minimal (empty) sieve – the complete false. Intermediate truth values correspond to situations when only part of events from $Past_e$ pertains to w. The truth value depends evidently on e, i.e. on the place and moment of assertion. In a similar manner for any subfunctor $\mathbf{F}$ from $\mathbf{Glob}$ the sieve defined by the map (26), is the local (from the view-point of an observer in e) truth value of the proposition «w is an element of $\mathbf{F}$». Note that we can look at assigning of truth values from a slightly another point. We can locally assign a sieve on e to any subfunctor $\mathbf{F}$ from $\mathbf{Glob}$:

$$[S_{\mathbf{F}}]_e =_{df} \{(e' \rightsquigarrow e) \in Mor_{\mathcal{C}}(\cdot, e) : \mathbf{F}_{e'} \neq \varnothing\}. \tag{32}$$

In this setting the local truth value (31) is identical to $[S_{\mathbf{w} \wedge \mathbf{Loc}}]_e$. Here the functor $\mathbf{w}$ appears such that $\mathbf{w}_e = \{w\}$ and all causal arrows are mapped onto identity of this one-element set.

The usefulness of the last notion let one assign a local truth-value sieve to assertions concerning space-time events without any reference to particular Belnap world. For example, from the view-point of an observer in e the assertion that «the event e_0 takes place»should intuitively be associated with the following sieve:

$$\{(e' \rightsquigarrow e) \in Mor_{\mathcal{C}}(\cdot, e) : \mathbf{Loc}_{e'} \cap \mathbf{Loc}_{e_0} \neq \varnothing\}. \tag{33}$$

One can note that this sieve is yielded by (32) for the presheaf $\mathbf{Loc} \wedge \mathbf{Loc}_{\mathbf{e_0}}$, where $\mathbf{Loc}_{\mathbf{e_0}}$ is the constant presheaf, so that $[\mathbf{Loc}_{\mathbf{e_0}}]_e = \mathbf{Loc}_{e_0}$ for any e. It is easy to see that the sieve (33) is maximal on e provided e_0 is compatible with e. On the contrary, (33) is the empty sieve if any event from $Past_e$ is incompatible with e_0. The truth value in e is intermediate if the considered assertion is absolutely true in only part (not all) of events from $Past_e$. In particular, this assertion is not completely false in any event if $\neg\mathbf{Loc} = \emptyset$, since in this case any world from $\mathbf{Loc}_{e_0}$ has non-empty intersection with the past of any other event. It is worth to note that the truth value of similar assertion from [2] is given by a co-sieve on e – by the set of all causal arrows from e to all common consequences of e and e_0. There last assertion states that «some consequence of the event e_0 will sooner or later be fixed in the memory of an e-localized observer». Meanwhile, this statement may be not absolutely true even if the sieve (33) is maximal.

5 Conclusion

Resuming, we see that the central object of our approach, the presheaf **Glob**, is made of Belnap's worlds – the main elements of branching space-time $\mathcal{C}$ considered as a partly-ordered set. The model of Heyting-value logic is realized on the set of subobjects of **Glob**. The known construction of subobject classifier let one assign to propositions the generalized truth values from the view-point of a local observer. The similarities can be traced between our approach and that of the work [2] which also uses the basis of topos theory. Nevertheless, there are significant differences. Not only events of space-time are considered but also their special collections, called Belnap's worlds. Because of this the emptiness of the presheaf $\neg\mathbf{Loc}$, as has been show, is equivalent to non-trivial relation between this two types of entities.

As a further development of the present approach we are going to present the categorial construction of local orthologics. This let us to make a step towards introducing quantum-like structures in the framework of branching space-time and, in perspective, towards a new line of reconciliation of classical and quantum conceptions.

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