

NATURE OF TIME AND PARTICLES-CAUSTICS: PHYSICAL WORLD IN ALGEBRODYNAMICS AND IN TWISTOR THEORY

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Abstract.

In the field theories with twistor structure particles can be identified with (spacially bounded) caustics of null geodesic congruences defined by the twistor field. As a realization, we consider the “algebrodynamical” approach based on the field equations which originate from noncommutative analysis (over the algebra of biquaternions) and lead to the complex eikonal field and to the set of gauge fields associated with solutions of the eikonal equation. Particle-like formations represented by singularities of these fields possess “elementary” electric charge and other realistic “quantum numbers” and manifest self-consistent time evolution including transmutations. Related concepts of generating “World Function” and of multivalued physical fields are discussed. The picture of Lorentz invariant light-formed aether and of matter born from light arises then quite naturally. The notion of the Time Flow identified with the flow of primordial light (“pre-Light”) is introduced in the context.

1 Introduction. The algebrodynamical field theory

Theoretical physics has arrived to the crucial point at which it should fully reexamine the sense and the interrelations of the three fundamental entities: fields, particles and space-time geometry. *String theory* offers a way to derive the low-energy phenomenology from the unique physics at Plankian scale. However, it doesn't claim to find the *origin* of physical laws, the *Code of Universe* and is in fact nothing but one more attempt to *describe* Nature (in a possibly the most effective way) but not at all to *understand* it.

Twistor program of R. Penrose [1, 2] suggests an alternative to string theory in the framework of which one can hope, in principle, to explain the origin of basic physical entities. For this, one only assumes the existence of the primary *twistor space* CP^3 which underlies the physical space-time and predetermines its Minkowsky geometry and, to some extent, the set of physical fields.

The most interesting manifestation of twistor structure is its ability to reduce the resolution of free massless (conformally invariant) equations (both linear and nonlinear ones, specifically of the Yang-Mills type) either to explicit integration in twistor space (the so called Penrose transform) or to resolution of purely algebraic problems (the Kerr theorem, the Ward construction etc. [2]). Making use of the Kerr theorem and of the Penrose's “nonlinear graviton construction”, one can also obtain, in a purely algebraic way, the whole set of the self-dual solutions to (complex) Einstein equations.

However, general concept of twistor program as a unified field theory is not at all clear or formulated up to now. Which equations are really fundamental, in which way can the massive fields be described and in which way the particles' spectrum can be obtained? And, finally, why precisely twistor, a rather refined mathematical object, should be taken as a basis of fundamental physics?

In the interim, twistor structure arises quite naturally in the so called *algebrodynamics* of physical fields which has been developed in our works. From general viewpoint, the paradigm of algebrodynamics can be thought of as a revive of Pithagorean or Platonean ideas about “*Numbers governing*”

physical laws". As the only (!) postulate of algebrodynamics one admits the existence of a certain unique and exceptional structure, of purely abstract (algebraic) nature, the internal properties of which completely determine both the geometry of physical space-time and the dynamics of physical fields (the latter being also algebraic in nature).

In the most successful realization of algebrodynamics principal structure of the "World algebra" has been introduced via generalization of complex analysis to exceptional noncommutative algebras of quaternion (\mathbb{Q}) type [15, 16, 17, 24, 26]. In particular, it was demonstrated that explicit account of noncommutativity in the very definition of functions "differentiable" in \mathbb{Q} inevitably results in the *non-linearity* of the generalized Cauchy-Riemann equations (GCRE) which follow. This makes it possible to regard the GCRE as fundamental dynamical equations of *interacting* physical fields represented by (differentiable) functions of the algebraic \mathbb{Q} -type variable.

A wide class of such fields-functions exists only for the complex extension of \mathbb{Q} -algebra, i.e. for the algebra of complex quaternions \mathbb{B} (*biquaternions*). Over the \mathbb{B} -algebra, the GCRE turn to be Lorentz invariant and acquire, moreover, the gauge and the spinor structures. On this base a self-consistent and unified *algebrodynamical field theory* has been constructed in our works [15, 16, 23, 24, 26, 25, 27, 29].

From the physical viewpoint, the most important property of GCRE is their direct correspondence to a fundamental *light-like* structure. The latter manifests itself in the fact that every (spinor) component $S(x, y, z, t) \in \mathbb{C}$ of the primary \mathbb{B} -field must satisfy the *complex eikonal equation* (CEE) [14, 15]

$$\eta^{\mu\nu} \partial_\mu S \partial_\nu S = (\partial_t S)^2 - (\partial_x S)^2 - (\partial_y S)^2 - (\partial_z S)^2 = 0, \quad (1)$$

where $\eta_{\mu\nu} = \text{diag}\{1, -1, -1, -1\}$ is the Minkowsky metric and ∂ stands for the partial derivative by respective coordinate. The CEE (1) is Lorentz invariant, nonlinear and plays the role similar to that of the *Laplace equation* in complex analysis. Each solution to GCRE can be reconstructed from a set of (four or less) solutions to CEE.

In the meantime, in [29] the intrinsic *twistor* structure of CEE has been discovered, and on its base the general solution of the nonlinear eikonal equation has been obtained. It was proved that, in this respect, every CEE solution belongs to one of two classes which both can be obtained from a twistor generating function via a simple and purely algebraic procedure. This construction allows also for definition of singular loci of the null geodesic congruences correspondent to the eikonal field – the *caustics*. Just at the caustics – the envelopes of the congruences – the neighbouring rays intersect each other, and the associated physical fields turn to infinity forming, thus, a unique *particle-like* object – a common source of the fields and of the congruence itself. Thus, in the algebrodynamical theory *the particles can be considered as (spacially bounded) caustics of the primordial null congruences*.

On the other hand, null congruences naturally define the universal local "transfer" of the basic twistor field with fundamental constant velocity "c" (in full analogy with the transfer of field by an electromagnetic wave) and point thus to exceptional role of the time coordinate in the algebrodynamical scheme and in twistor theory in general. Existence of the "Flow of Time" becomes therein a direct consequence of the existence of Lorentz invariant "aether" formed by the primordial light-like congruence ("preLight"). In the paper, we underline the principal property of *multivaluedness* of the fundamental complex solution to CEE ("World solution") and of the physical fields associated with it. As a result, at each space-time point one has a *superposition* of a great number of rays which belong to locally distinct null congruences, and the Time Flow turns to be *multi-directional*, i.e. consists of a number of superposed "subflows" (linked globally by complex structure into a unique physical "corpuscular-field" dualistic complex).

In section 2 we consider the twistor structure of CEE and the procedure of algebraic construction of its two classes of solutions. A few simple illustrative examples are presented. In section 3 we discuss the caustic structure of the CEE solutions, in particular of spatially bounded type (particle-like singular objects), and the properties of associated physical fields. In section 4, we introduce

the “World function” responsible for generation of the “World solution” to CEE and discuss the related concept of multivaluedness of physical fields. Final section 5 is devoted to some general issues which bear on the nature of physical time. The notions of the *primordial light* (“pre-Light”) and of the *light-formed aether* are introduced, and the Time Flow is actually identified with the Flow of preLight. Intrinsic structure of these fundamental flows is studied which relates to the property of multivaluedness of the basic twistor field.

The article is a continuation of our paper [41]. In order to simplify the presentation, we avoid to apply the 2-spinor and the other refined mathematical formalisms, for this referring a prepared reader to our recent papers [24, 26, 27, 29].

2 The two classes of solutions to the complex eikonal equation

The eikonal equation describes the process of propagation of wave fronts (field discontinuities) in any relativistic theory, in Maxwell electrodynamics in particular [4, 5]. Physical and mathematical problems related to the eikonal equation were dealt with in a lot of works, see e.g. [6, 7, 8, 10, 11, 12].

The complex eikonal equation (CEE) arises naturally in problems of propagation of restricted light beams [13] and in theory of congruences related to solutions of Einstein or Einstein-Maxwell system of equations [14]. We, however, interpret the complex eikonal, to the first turn, as a fundamental physical field which describes, in particular, the interacting and “self-quantized” particle-like objects formed by singularities of the CEE solutions. By this, the electromagnetic and the other conventional physical fields can be associated with any solution of the CEE; they are responsible for description of the process of interaction of particles-singularities. Note that particle-like properties of field singularities related to the *5-dimensional* real eikonal field have been studied in [9]; the concept of particles as singularities of electromagnetic and eikonal fields has been incidentally discussed by many authors, in particular by H. Bateman [6] as far as in 1915.

We start with a definition, together with Cartesian space-time coordinates $\{t, x, y, z\}$, of the so called *spinor* or *null* coordinates $\{u, v, w, \bar{w}\}$ (the light velocity is taken to be unity, $c = 1$)

$$u = t + z, \quad v = t - z, \quad w = x - iy, \quad \bar{w} = x + iy \quad (2)$$

which form the *Hermitian* 2×2 matrix $X = X^+$ of coordinates

$$X = \begin{pmatrix} u & w \\ \bar{w} & v \end{pmatrix} \quad (3)$$

In the spinor coordinates representation the CEE (1) looks as follows:

$$\partial_u S \partial_v S - \partial_w S \partial_{\bar{w}} S = 0. \quad (4)$$

The CEE possesses a remarkable *functional invariance* [15, 16]: for every $S(X)$ being its solution any (differentiable) function $f(S(X))$ is also a solution. The eikonal equation is known also [6] to be invariant under transformations of the full 15-parameter *conformal group* of the Minkowsky space-time.

Let us take now an *arbitrary homogeneous* function Π of two pairs of complex variables $\{\xi, \tau\}$

$$\Pi = \Pi(\xi_0, \xi_1, \tau^0, \tau^1) \quad (5)$$

which are *linearly dependent* at any space-time point via the so called *incidence relation*

$$\tau = X\xi \Leftrightarrow \tau^0 = u\xi_0 + w\xi_1, \quad \tau^1 = \bar{w}\xi_0 + v\xi_1, \quad (6)$$

and which transform as *2-spinors* under Lorentz rotations¹. The pair of 2-spinors $\{\xi(X), \tau(X)\}$ linked through Eq.(6) is known as a (null projective) *twistor* of the Minkowsky space-time [2].

Let us assume now that one of the components of the spinor $\xi(X)$, say ξ_0 , is not zero. Then, by virtue of homogeneity of the function Π , we can reduce the number of its arguments to *three projective twistor variables*, namely to

$$\Pi = \Pi(G, \tau^0, \tau^1), \quad G = \xi_1/\xi_0, \quad \tau^0 = u + wG, \quad \tau^1 = \bar{w} + vG \quad (7)$$

Now we are in order to formulate the main result proved in our paper [29].

Theorem. *Any (analytical) solution of CEE belongs, with respect to its twistor structure, to one of two and only two classes and can be obtained from some generating twistor function of the form (7) via one of two simple algebraical procedures (described below)*

(note only that for solutions with zero spinor component, $\xi_0 = 0$, another gauge, in compare with the one used above, should be choosed).

To obtain the first class of solutions, let us simply resolve the algebraic equation defined by the function (7)

$$\Pi(G, u + wG, \bar{w} + vG) = 0 \quad (8)$$

with respect to the only unknown G . In this way we come to a complex field $G(X)$ which necessarily satisfies the CEE. Indeed, after substitution $G = G(X)$ Eq.(8) becomes an *identity* and, in particular, can be differentiated with respect to the spinor coordinates u, v, w, \bar{w} . Then we get

$$P\partial_u G = -\Pi_0, \quad P\partial_w G = -G\Pi_0, \quad P\partial_{\bar{w}} G = -\Pi_1, \quad P\partial_v G = -G\Pi_1, \quad (9)$$

where Π_0, Π_1 are the partial derivatives of Π with respect to its twistor arguments τ^0, τ^1 while P is its *total* derivative with respect to G ,

$$P = \frac{d\Pi}{dG} = \partial_G \Pi + w\Pi_0 + v\Pi_1, \quad (10)$$

which we thus far assume to be nonzero in the space-time domain considered. Multiplying then Eqs.(9) we prove that $G(X)$ satisfies the CEE in the form (4). It is easy to check that *arbitrary* twistor function $S = S(G, u + wG, \bar{w} + vG)$, under substitution of the obtained $G = G(X)$, also satisfies the CEE (owing to the functional constraint (8) it depends in fact on only *two* of three twistor variables).

To obtain the second class of CEE solutions, we have from the very beginning to differentiate the function Π with respect to G and only after this to resolve the resulting algebraic equation

$$P = \frac{d\Pi}{dG} = 0 \quad (11)$$

with respect to G again. Now the function $G(X)$ does not satisfy the CEE; however, if we substitute it into (7) the quantity Π becomes an explicit function of space-time coordinates and necessarily satisfies the CEE (as well as any function $f(\Pi(X))$ by virtue of functional invariance of the CEE). Indeed, differentiating the function Π with respect to the spinor coordinates we get

$$\partial_u \Pi = \Pi_0 + P\partial_u G, \quad \partial_w \Pi = G\Pi_0 + P\partial_w G, \quad \partial_{\bar{w}} \Pi = \Pi_1 + P\partial_{\bar{w}} G, \quad \partial_v \Pi = G\Pi_1 + P\partial_v G, \quad (12)$$

and, taking into account the generating condition (11), we immediately find that the function Π itself obeys the CEE (4).

¹To simplify the notation, we do not distinguish the primed and unprimed spinor indices. In the incidence relation (6) the standard factor "i" (imaginary unit) is omitted what is admissible under the proper redefinition of the twistor norm

The functional condition (8) and, therefore, the CEE solutions of the first class are in fact well known. Indeed, apart from the CEE, the field $G(X)$, if it is obtained by the resolution of Eq.(8), satisfies (as it is easily seen from Eqs.(9) for derivatives), the *over-determined* system of differential constraints

$$\partial_u G = G\partial_w G, \quad \partial_{\bar{w}} G = G\partial_v G \quad (13)$$

which define the so called *shear-free (null geodesic) congruences* (SFC). By this, algebraic Eq.(8) represents (in implicit form) *general solution* of Eqs.(13), i.e. describes the whole set of SFC in the Minkowski space-time. This remarkable statement proved in [18] is known as the *Kerr theorem*.

The second class of CEE solutions generated by algebraic constraint (11), to our knowledge, hasn't been considered in literature previously². It is known, however, that condition (11) defines the *singular locus* for SFC, i.e. for the CEE solutions obtained from the Kerr constraint (8). Precisely, condition (11) fixes the *branching points* of the principal complex field $G(X)$ or, equivalently, – the space-time points where Eq.(8) has *multiple roots*. As to the CEE solutions of the second class themselves, their branching points occur at the locus defined by another condition which evidently follows from generating Eq.(11) and has the form

$$\Lambda = \frac{d^2\Pi}{dG^2} = 0. \quad (14)$$

The null congruences (especially the congruences with zero shear), as well as their singularities and branching points, play crucial role in the algebrodynamical approach. They will be discussed below in more details. Here we only repeat that, as it has been proved in [29], *the two simple generating procedures described above exhaust all the (analytical) solutions to the CEE representing, thus, its general solution*.

The obtained result can be thought of as *a direct generalization of the Kerr theorem*. Below, in order to make the exposition more clear, we present several examples of the described construction.

1. Static solutions. Let the generating function Π depends on its twistor variables in the following way:

$$\Pi = \Pi(G, H), \quad H = G\tau^0 - \tau^1 = wG^2 + 2zG - \bar{w}, \quad (15)$$

where $z = (u - v)/2$, and the time coordinate $t = (u + v)/2$ is, in this way, eliminated. It is evident that the generating ansatz (15) covers the whole class of *static* CEE solutions.

In [21, 14] it was proved that static solutions to the SFC equations (and, therefore, static solutions to the CEE too) with *spacially bounded* singular locus are exhausted, up to 3D translations and rotations, by the *Kerr solution* [18] which follows from generating function of the form

$$\Pi = H + 2iaG = wG^2 + 2z^*G - \bar{w}, \quad (z^* = z + ia) \quad (16)$$

with a real constant parameter $a \in \mathbb{R}$. Explicitly resolving equation $\Pi = 0$ which is quadratic in G we obtain the two “modes” of the field $G(X)$

$$G = \frac{\bar{w}}{z^* \pm r^*} = \frac{x + iy}{z + ia \pm \sqrt{x^2 + y^2 + (z + ia)^2}} \quad (17)$$

which in the case $a = 0$ correspond to the ordinary *stereographic projection* $S^2 \mapsto \mathbb{C}$ from the North or the South pole respectively. It is easy to check that this solution and also its twistor counterparts

$$\tau^0 = t + r^*, \quad \tau^1 = G\tau^0, \quad (18)$$

²Study of solutions of the *real* eikonal equation by differentiation of generating functions depending on coordinates as parameters is used in general theory of singularities of caustics and wavefronts [11]

satisfy the CEE (as well as any function of them). Correspondent SFC is in the case $a = 0$ *radial* with a point singularity; in general case $a \neq 0$ the SFC is formed by the rectilinear constituents of a system of hyperboloids and has a *ring-like* singularity of a radius $R = |a|$. Using this SFC, a Riemannian metric (of the “Kerr-Schild type”) and an electric field can be defined which satisfy together the electrovacuum Einstein-Maxwell system. In the case $a = 0$ this is the Reissner-Nordström solution with Coulomb electric field, in general case – the Kerr-Newman solution with three characteristic parameters: the mass M , the electric charge Q and the angular momentum (spin) Mca , – for which the field distribution possesses also the proper magnetic moment Qa which corresponds to the gyromagnetic ratio specific for the Dirac particle [19, 20]. In the algebrodynamical scheme, moreover, *electric charge of the point or the ring singularity is necessarily fixed in modulus*, i.e. “elementary” [15, 16, 27, 28] (see also [24] where a detailed discussion of this solution in the framework of algebrodynamics can be found).

Now let us obtain, from the same generating function, a solution to CEE of the second class. Differentiating Eq.(16) with respect to G and equating derivative to zero, we get $G = -z^*/w$ and, substituting this expression into Eq.(16), obtain finally the following solution to CEE (which is univalued everywhere on 3D-space):

$$\Pi = -\frac{(r^*)^2}{w} = -\frac{x^2 + y^2 + (z + ia)^2}{x - iy}. \quad (19)$$

It is instructive to note that equation $\Pi = 0$, being equivalent to two real-valued constraints $z = 0$, $x^2 + y^2 = a^2$, defines here the ring-like singularity for the Kerr solution (17), as it should be in account of the theorem above presented (for this, see also section 4).

Static solutions of the II class with spatially bounded singularities are not at all exhausted by the solution (19). Consider, for example, solutions generated by the functions

$$\Pi = \frac{G^n}{H}, \quad n \in \mathbb{Z}, \quad n > 2. \quad (20)$$

We’ll not write out correspondent solutions in explicit form and shall restrict ourselves by examination of the spacial structure of their singularities which can be obtained from the joint system of equations $P = 0$, $\Lambda = 0$, see Eqs.(11),(14). Eliminating from the latter the unknown field G we find that singularities (branching points of the eikonal field) have again the ring-like form $z = 0$, $x^2 + y^2 = R^2$ with radii equal to

$$R_n = \frac{a(n-1)}{\sqrt{n(n-2)}} \quad (21)$$

The cases $n = 1, 2$ evidently need special consideration. For $n = 1$ equating to zero derivative of the function G/H we find $G = \pm i\bar{w}/\rho$ with $\rho = \sqrt{x^2 + y^2}$. This brings us after substitution to the following solution of the CEE:

$$\Pi = (z + ia \pm i\sqrt{x^2 + y^2})^{-1} \quad (22)$$

which has the pole at the ring $z = 0$, $x^2 + y^2 = a^2$ but *has a branching point only on the origin $r = 0$, i.e. under any a corresponds to the point singularity.*

In the case $n = 2$ via analogous procedure we get $G = \bar{w}/z^*$ and after substitution come to the following solution of the CEE [29]:

$$\Pi = \frac{\bar{w}}{r^*} = \frac{x + iy}{x^2 + y^2 + (z + ia)^2} \quad (23)$$

which is of the same structure as (the inverse of) the solution (19). As the latter, it has no branching points on the real space-time slice while its pole corresponds to the Kerr ring. Let us take for simplicity

$a = -1$; then solution (23) can be rewritten in the following familiar form:

$$\Pi = i \frac{x + iy}{2z + i(r^2 - 1)} \quad (24)$$

which can be easily identified as the standard *Hopf map*. As the solution of the CEE it has been studied in the recent paper [22] where its geometrical and topological nature has been examined in detail. We suspect also that generalized Hopf maps considered therein and obeying the CEE are closely related (in the case $m = 1$) to the CEE solutions generated by the functions (20) and, as the latters, has the ring singularities correspondent to those represented by Eq.(21). However, this should be verified by direct calculations. Note, finally, that trivial generalization of the functions (20) $\Pi = G^n/H^m$, $n, m \in \mathbb{Z}$ gives rise to a set of CEE static solutions with two integers which correspond, perhaps, to those introduced in [22].

Wave solutions. Consider also the class of generating functions dependent on one of the two twistor variables τ^0, τ^1 only, say on τ^0 :

$$\Pi = \Pi(G, \tau^0) = \Pi(G, u + wG). \quad (25)$$

Both classes of the CEE solutions obtained via functions (25) will then depend on only two spinor coordinates $u = t + z$, $w = x - iy$. This means, in particular, that the fields propagate along the Z -axis with fundamental (light) velocity $c = 1$. A “photon-like” solution of this type, with singular locus spacially bounded in all directions, was presented in [28].

Notice also that an example of the CEE solution with a considerably more rich and realistic structure of singular locus is presented below in section 4 (see also [28]).

3 Particles as caustics of the primordial light-like congruences

It’s well known that a null congruence of rays corresponds to any solution of the eikonal equation; it is orthogonal to hypersurfaces of constant eikonal $S = const$ and directed along the 4-gradient vector $\partial_\mu S$. Usually, these two structures define the *characteristics* and *bicharacteristics* of a (linear) hyperbolic-type equation, e.g. of the wave equation $\square\Psi = 0$.

In the considered complex case, i.e. in the case of CEE, the hypersurfaces of constant eikonal and the 4-gradient null congruences belong geometrically to the *complex extension* $\mathbb{C}M^4$ of the Minkowski space-time which looks here quite natural in account of the complex structure of the primary bi-quaternion algebra \mathbb{B} . The problem of physical sense of the additional (imaginary) dimensions is much important and nontrivial, and we hope to discuss it in the forthcoming paper.

Here we use another interesting property: existence of a null geodesic congruence defined on a *real* space-time for every of the *complex-valued* solutions to CEE. This remarkable property follows directly from the twistor structure inherent to CEE. Indeed, according to the theorem above-presented, any of the CEE solutions (both of the I and the II classes) is fully determined by a (null projective) twistor field $\{\xi(X), \tau(X)\}$ (in the choosed gauge one has $\xi_0 = 1$; $\xi_1 = G(x)$) subject to the incidence relation (6). This latter “Penrose equation” can be explicitly resolved with respect to the space coordinates $\{x_a, a = 1, 2, 3\}$ as follows:

$$x_a = \frac{\Im(\tau^+ \sigma \xi)}{\xi + \xi} - \frac{\xi^+ \sigma \xi}{\xi + \xi} t, \quad (26)$$

with $\{\sigma_a\}$ being the Pauli matrices and the time t remaining a *free parameter*. Eq.(26) manifests that the primordial spinor field $\xi(X)$ *reproduces its value along the 3D rays formed by the unit “director vector”*

$$\vec{n} = \frac{\xi^+ \vec{\sigma} \xi}{\xi + \xi}, \quad \vec{n}^2 = 1, \quad (27)$$

and propagates along these locally defined directions with fundamental constant velocity $c = 1$. In the choosed gauge we have for Cartesian components of the director vector (27)

$$\vec{n} = \frac{1}{(1 + GG^*)} \{(G + G^*), -i(G - G^*), (1 - GG^*)\}, \quad (28)$$

the two its real degrees of freedom being in one-to-one correspondence with the two components of the complex function $G(X)$.

Thus, for every solution of the CEE the space is foliated by a congruence of *rectilinear* light rays, i.e. by a *null geodesic*³ congruence (NGC). Notice that the director vector obeys the *geodesic equation* [41]

$$\partial_t \vec{n} + (\vec{n} \vec{\nabla}) \vec{n} = 0. \quad (29)$$

The basic field $G(X)$ of the NGC can be always extracted from one of the two algebraic constraints (8) or (11) which at any space-time point possess, as a rule, not one but rather a *finite (or even infinite) set* of different solutions. Suppose that generating function Π is *irreducible*, i.e. can't be factorized into a number of twistor functions of the same structure (otherwise, we should make a choice in favour of one of the multiplies). Then a generic solution of the constraints will be nothing but a *multivalued complex function* $G(X)$. Choose locally (in the vicinity of a particular point X) one of the continious *branches* of this function. Then a particular NGC and a set of physical fields can be associated with this branch, i.e. with one of the “modes” of the multivalued field distribution.

Specifically, for any of the I class CEE solutions the spinor $F_{(AB)}$ of *electromagnetic field* can be defined explicitly in terms of twistor variables of the solution [26, 27, 28]:

$$F_{(AB)} = \frac{1}{P} \left\{ \Pi_{AB} - \frac{d}{dG} \left(\frac{\Pi_A \Pi_B}{P} \right) \right\}. \quad (30)$$

where Π_A, Π_{AB} are the first and the second order derivatives of the generating function Π with respect to its two twistor arguments τ^0, τ^1 . For every branch of the solution $G(X)$ *this field locally satisfies Maxwell homogeneous (“vacuum”) equations*. Moreover, as it has been demonstrated in [16, 23, 26], a complex-valued $SL(2, \mathbb{C})$ *Yang-Mills field* and a *curvature field* (of some effective Riemannian metric) can be also defined through only the same principal function $G(X)$ for any of the CEE solution of the first class.

Consider now analytical continuation of the function $G(X)$ up to one of its branching points which corresponds to a multiple root of Eq.(8) (or, alternatively, of Eq.(11) for solutions of the II class). At this point $P = 0$, and the strength of electromagnetic field (30) turns to infinity. The same holds for the other associated fields, for curvature field⁴ in particular [21]. Thus, the locus of branching points (which can be 0-, 1- or even 2-dimensional, see section 4) manifests itself as a *common source* of a number of physical fields and can be identified (at least, in the case when it is bounded in 3-space) as a unique *particle-like* object.

Such formations are capable of much nontrivial evolution simulating physical interactions or even mutual *transmutations* represented by *bifurcations* of the field singularities (see, e.g., the example in section 4). They possess also a realistic set of “quantum numbers” including a *self-quantized electric charge* and a *Dirac-type gyromagnetic ratio* (equal to that for a spin 1/2 fermion) [19, 20, 24]. Numerous examples of such solutions and their singularities can be found in our works [23, 24, 25, 26].

On the other hand, for the light-like congruences - NGC - associated with CEE solutions via the guiding vector (28) the locus of branching points coincides with that of the principal G -field and represents the familiar *caustic* structure, i.e. the envelope of the system of rays at which the

³On the flat Minkowsky background the geodesics are evidently rectilinear

⁴Associated Yang-Mills fields possess, generically, additional *string-like* singularities

neighbouring rays intersect each other (“focusize”). From this viewpoint, within the algebrodynamical theory the “particles” are nothing but the caustics of null rectilinear congruences associated with the CEE solutions.

4 The World function and the multivalued physical fields

At this point we have to decide which of the two types of the CEE solutions can be in principle taken in our scheme as a representative for description of the Universe structure as a whole. As a “World solution” we choose a CEE solution of the first class because a lot of peculiar geometrical structures and physical fields can be associated with any of them [16, 24, 26]. Such a solution can be obtained algebraically from the Kerr functional constraint (8) and a generating twistor “World function” Π which is *exceptional* with respect to its internal properties; geometrically it gives rise to an NGC with a special property - zero shear [2, 3].

Moreover, a *conjugated* CEE solution of the II class turns then also to be involved into play since it defines a characteristic hypersurface of the (I class) “World solution”. In fact, this is determined as a solution of the joint algebraic system of Eqs.(8),(11). Precisely, if we resolve Eq.(11) with respect to G and substitute the result into (8), equation $\Pi(G(X)) = 0$ would define then the singular locus (the characteristic hypersurface) of the World solution. On the other hand, the function $\Pi(G(X))$ would necessarily satisfy the CEE representing its II class solution in account of the theorem presented in section 2. Thus,

the eikonal field here carries out two different functions being a fundamental physical field (as a CEE solution of the I class) and, at the same time, a characteristic field (as a solution of the II class) which describes the locus of branching points of the basic field (i.e., the discontinuities of its derivatives).

Let us conjecture now that the World function Π is an *irreducible polinomial of a very high but finite order*⁵ so that Eq.(8) is an algebraic (not a transcendental) one. Note that in this case Eq.(8) defines an algebraic surface in the projective twistor space $\mathbb{C}P^3$.

The World solution consists then of a finite number of modes – branches of multivalued complex G -field. A finite number of null directions (represented in 3-space by the director vector (28)) and an equal number of locally distinct NGC would exist then *at every point*.

Any pair of these congruences at some fixed moment of time will, generically, has an envelope consisting of a number of connected one-dimensional components-caustics⁶. Just these spacial structures (in the case they are bounded in 3-space) represent here the “particles” of generic type. Other types of particle-like structures are formed at the focal points of *three or more* NGC where Eq.(8) has a root of higher multiplicity. Formations of the latter type would, of course, meet rather rarely, and their stability is problematic. One can speculate on their possible relation to particle’s excitations – *resonances*.

Nonetheless, we can model both types of particles-caustics in a simple example based on generating twistor function of the form [28]

$$\Pi = G^2(\tau^0)^2 + (\tau^1)^2 - b^2G^2 = 0, \quad b = \text{const} \in \mathbb{R}, \quad (31)$$

which leads to the 4-th order polinomial equation for the G -field. At initial moment of time $t = 0$, as it can be obtained analytically, the singular locus consists of a pair of point singularities (with opposite

⁵This conjecture is, in fact, not at all necessary. Indeed, one can easily imagine that the World function leads to the Kerr Eq.(8) which possesses an **infinite number of roots** for complex-valued field function $G(X)$ at any space-time point X

⁶In fact, the caustics of **generic type** are determined by one complex condition $\Pi(G(X)) = 0$ (i.e., by two real equations) on three coordinates and, at a fixed moment of time $t = t_0$, correspond to a number of one-dimensional curves (“strings”)

and equal in modulus “elementary” electric charges) and of a neutral 2-surface (ellipsoidal *cocoon*) covering the charges (see [28] for more details). The latter corresponds to the intersection of all of the 4 modes of the multivalued solution while each of the point charges is formed by intersection of a particular pair of (locally radial, Coulomb-like) congruences [28]. The time evolution of the solution and of its singularities is very peculiar: for instance, at $t = b/\sqrt{2}$ the point singularities cancel themselves at the origin $r = 0$ simulating thus the process of *annihilation* of elementary particles. Moreover, this process is accompanied by emission of the *singular light-like wavefront* represented by another 2-dimensional component of connection of the caustic structure.

Thus, we see that the multivalued fields are quite necessary for to ensure the self-consistent structure and evolution of a complicated (realistic) system of particles - singularities. One only should not be confused by such, much unusual, property of the principal G -field and, especially, by multivalued nature of the other associated fields including the electromagnetic one.

Indeed, in convenient classical theories, the fields are in fact only a tool which serves for adequate description of particle dynamics (including the account of retardation etc.) and for nothing else. In nonlinear theories, as well as in our algebrodynamical scheme, the fields are moreover responsible for *creation* and structure of particles themselves, as regular *solitons* or *singularities* of fields respectively. In the first, more familiar case we, apparently, should consider the fields to be univalued. The same situation occurs in the framework of quantum mechanics where the quantization rules often follow from the requirement for the wave function to be univalued.

However, as we have seen above, in the algebrodynamical construction *the field distributions must not necessarily be univalued !* On the other hand, acceptance of fields’ multivaluedness does not at all prevent to obtain the discrete spectrum of characteristics in a full analogy with quantum mechanics. For example, the requirement of univaluedness of a **particular, locally choosed mode** of the principal G -field and of the associated electromagnetic field (far from the branching points of the first and, consequently, from the infinities of the second!) leads to the general property of *quantization of electric charge of singularities* in the framework of algebrodynamical theory [27, 28].

As to the process of “measurement” of the field strength, say, of electromagnetic field, it directly relates to only the measurements of particles’ accelerations, currents etc., and only after the measurements the results are translated into conventional field language. However, this is not at all necessary (in recall, e.g, of the Wheeler-Feynman electrodynamics and of numerous “action-at-a-distance” approaches [30, 31]). In fact, “we never deal with fields but only with particles” (F. Dyson).

In particular, on the classical (nonstochastic) level we can deal, effectively, with the *mean value* of the set of field modes at a point; similar concept based on purely quantum considerations has been recently developed in the works [32]. In our scheme, the true role of the multivalued field will become clear only after the spectrum and the effective mechanics of particles-singularities will be obtained in a general and explicit form.

We hope that a sort of psychological barrier for acceptance of general idea of the field multivaluedness will be get over as it was with possible *multidimensionality* of physical space-time. The advocated concept seems indeed very natural and attractive. In the purely mathematical framework, multivalued solutions of PDEs are the most common in comparison with the familiar δ -type distributions [33, 8]. From physical viewpoint, this makes it possible to naturally define a dualistic “corpuscular-field” complex of a very rich structure which, actually, gathers all the particles in the Universe into a unique object. The caustics-singularities are well-defined themselves and undergo a collective self-consistent motion free of any ambiguity or divergence (the latters can arise here only in result of incorrect description of the evolution process and can be removed, if arise, on quite legal grounds, contrary, say, to the renormalization procedure in the quantum field theory). Note also that recently accomplished universal local classification of singularities of differentiable maps, in particular of caustics and wavefronts [11], can explicitly bear on the characteristics of elementary particles if the latters are treated

in the framework of the algebrodynamical theory.

As to the principal problem of the choice of a particular representative of the generating *World function* Π of the *Universe* we are ready to offer an interesting candidature being in hope to discuss it elsewhere.

5 The light-formed relativistic aether and the nature of time

Light-like congruences (NGC) are the basic elements of the picture of physical world which arises in the algebrodynamical scheme and, to some extent, in twistor theory in general. The rays of the NGC densely fill the space and consist of a great number of branches - components superposed at each space point and propagating in different directions with constant in modulus and universal (for any branch of multivalued solution, any point and any system of reference) fundamental velocity. *There is nothing in the Universe except this primordial light flow ("pre-Light Flow") because the whole Matter is born by pre-Light and from pre-Light at the caustic regions of "condensation" of the pre-light rays.*

In a sense, one can speak here about an exceptional form of *relativistic aether* which is formed by a flow of pre-Light. Such an exceptional form of the World aether has nothing in common with old models of the *light-carrying* aether which had been considered as a sort of elastic medium. Here, the *light-formed* aether consists of structureless "light elements" and is, obviously, in full correspondence with special theory of relativity ⁷.

At the same time, notions of the aether formed by pre-Light and of the matter formed by its "thickenings" evoke numerous associations with the Bible and with ancient Eastern philosophy. Certainly, there were theologians, philosophers or mystics who were brought to imagine a similar picture of the World. However, in the framework of successive physical theory this picture becomes more trustworthy and, to our knowledge, has not been yet discussed in literature ⁸.

On the other hand, existence of the primordial light-formed aether and manifestation of universal property of local "transfer" of the *aether - generating field* $G(X)$ with constant fundamental velocity $c = 1$ points to different status of space and time coordinates and *offers a new approach to the problem of physical time* as a whole. By this, it is noteworthy that since in 1908 H. Minkowsky has joined space and time into a unique 4-dimensional continuum, no further understanding of the nature of time has been achieved in fact. Moreover, this synthesis has "shaded" the principal distinction of space and time entities and clarified none of such problems as (micro/macro)irreversibility, (in)homogeneity and (non)locality of time, its dependence on material processes etc.

In the interim, the key problem of Time can be formulated in a rather simple way. *Subjectively*, we perceive time as a continuous intrinsic motion, a latent *flow*. Everybody comprehends in a moment, as the ancient Greeks did, what is meant by the "River of Time", the "Flow of Time". As a rule, we consider this intrinsic motion to be independent on our will and on material processes and uniform: not for nothing, in physics the flow of time is modelled by the uniform motion of, say, the record tape etc. Moreover, under variations in time one does not only observe the *conservation* of a particular set of *integral* quantities (which is widely used in the orthodox physics) but perceives subjectively the complete *repetition, reproduction of the local states of any system*; that's why for measurements of time itself we use *clocks* whose principle of operation is based on reproducible, periodical processes. In other words, whereas one has much ambiguous and diverse distributions of spacial positions of

⁷At present, it seems rather strange that A. Einstein didn't come himself to the concept of relativistic aether so consonant with the ideas of STR and with his favourite *Mach principle*. Suprisingly, R. Penrose also overlooked this opportunity which follows naturally from his twistor theory

⁸Similar in some aspects ideas have been advocated in the works [35, 37, 38]. Note, in particular, the concept of the "radiant particle" offered by L.S. Shikhobalov [34]

physical bodies, all they and we all have always *one and the same monotonically increasing time coordinate*, i.e. are in a common and permanent motion together with the “Time River”.

Surprisingly, almost all these considerations are absent in the structure of theoretical physics and, in particular, in relativity theory. To bring into correspondence the results of calculations with practice (e.g. for the Cauchy problem etc.) one chooses a “time orthogonal hypersurface”, i.e. quite ambiguously fixes the unity of the present moment of time, of the moment “now”, perceived subjectively by everybody; however, *there are no intrinsic reasons for this choice in the very structure of theoretical physics, including the STR.*

At least partially, such a situation is caused by the following. The notion of everywhere existing, eternal Flow of Time immediately leads to the problem of its (material? pre-material?) carrier. In this connection, the works of N.A. Kozyrev [39] should be marked, of course, in which the concept of the “active” Flow of Time influencing directly the material processes has been proposed. To our opinion, however, there are no reliable physical grounds at present which confirm the Kozyrev’s ideas, and no mechanism of “interaction” of this exotic form of matter with the ordinary ones. As to the algebrodynamical paradigm, the Time Flow is non-material therein: it does not *interact* or *influence* the Matter at all but just *forms* it. In distinction from the Kozyrev’s concept, we do not deal here with various material entities only one of them being the Time itself: on the contrary, here we have one triply-unique entity – preLight-Time-Matter ⁹.

On the other hand, under consideration of the problem of the carrier of the Time Flow, we inevitably return back to the notion of some form of the *World aether* which has been exiled from physics after the triumph of Einstein’s theory. To do without aether, none Flow of Time can be successively included into the structure of theoretical physics and none subjectively perceived properties of time can be precisely formulated and described.

However, in a paradoxical way, just the STR with its postulate of the invariance of light velocity justifies the introduction of the *dynamical Lorentz invariant aether* formed by the light-like congruences as the primary element of physical World. Specifically, the Time Flow can be naturally identified now with the Flow of Primordial Light (pre-Light), and the “River of Time” turns to be nothing but the “River of Light”. Moreover, it is the universality of light velocity which explains our subjective perception of uniformity and homogeneity of the Time Flow.

There is, however, another, the most striking and unexpected feature of the introduced concept of physical time. *The Time Flow manifests here itself as a superposition of a great number of distinctly directed and locally independent components - “subflows”.* At any point of 3-dimensional space there exists a (finite) set of directions: each mode of the primordial multivalued field $G(X)$ defines one of these directions and *propagates (reproduces its value) along it* forming thus one of the constituents of the (globally unique) Flow of pre-Light identical to the Flow of Time.

One can conjecture that just by virtue of the local multivaluedness we are not capable of to perceive the particular local *direction* of the Time Flow. Apart from this, it is natural to assume that in the tremendously complicated structure of the World solution a *stochastic* component is necessarily present, particularly in the structure of the primordial Light-Time Flow. This results in chaotic variations of local directions of the light-like congruences which are certainly inaccessible for perception. On the other hand, it is the existence of (constant in modulus and the same for all of the branches of the multivalued World solution) *fundamental propagation velocity of the pre-Light rays* which makes it possible to feel the Flow of Time in general and to subjectively regard it as uniform and homogeneous in particular.

⁹Much more close the approach turns to be to the concept of “Time-generating Flows” developed by A.P. Levich [40]. Projective structure of a specific type closely related to twistors has been applied for explanation of the time nature in the concept of M. Saniga [36]

6 Conclusion

Thus, we have examined the realization of the algebrodynamical approach in which as a base of unified physical theory one only structure of a purely abstract nature is chosen, namely the algebra of complex quaternions and the generalized CR-equations – the conditions of differentiability in this algebra. Very the same structure can be successively expressed, in fact, on a number of equivalent geometrical languages (covariantly constant fields, twistor geometry, shear-free congruences etc.).

Primary GCR-equations result directly in the field of complex eikonal regarded in theory as a fundamental physical field (alternative in a sense to the linear fields of quantum mechanics). In its turn, the eikonal field is here closely related to the fundamental 2-spinor and twistor fields, on whose language, in particular, the general solution of the complex eikonal equation is formulated. Through the eikonal field also the other ones are defined, namely the electromagnetic and Yang-Mills fields. Singularities of the eikonal and of correspondent null congruences are considered as particle-like formations (“self-quantized” and effectively interacting).

In result, physical picture of the World which arises as a consequence of one only algebraic structure appears as very beautiful and unexpected. As its basic elements it contains the primordial light flow – “pre-Light” – and the relativistic aether formed by the latter, multivalued physical fields and prelight-born matter (consisting of particles-caustics formed by the superposition of individual branches of the unique pre-light congruence in the points of their “focusization”).

As very natural and deep seems to be the arising in theory connection between the existence of universal velocity (velocity of “light”) and of the time flow; connection which permits to understand, in a sense, the origin of the Time itself. *Time is nothing but the primordial Light*; these two entities are undividable. On the other hand, *there is nothing in the World except the preLight Flow* which gives rise to all the “dense” Matter in the Universe.

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References

- [1] R. Penrose, in: *Quantum gravity: an Oxford Symposium*, eds. C.J. Isham, R. Penrose, D.W. Sciama. – Clarendon Press, Oxford, 1975.
- [2] R. Penrose and W. Rindler, “Spinors and Space-Time”. Vol.2. – Cambridge Univ. Press, Cambridge, 1986.
- [3] R. Penrose, *Classical and Quantum Gravity*, **14**, A299 (1997).
- [4] V.A. Fock, “Theory of Space, Time and Gravitation”. – “GITTL”, Moscow, 1955 (in Russian).
- [5] A.Z. Petrov, “New Methods in General Relativity”. – “Nauka”, Moscow, 1966 (in Russian).
- [6] H. Bateman, “The Mathematical Analysis of Electrical and Optical Wave-Motion”, Dover Publ. Inc., 1955.

- [7] S. Frittelli, E.T. Newman and G. Silva-Ortigoza, *J. Math. Phys.*, **40**, 383 (1999);
E.T. Newman and A. Perez, *J. Math. Phys.*, **40**, 1089, 1999.
- [8] F.Lizzi, G.Marmo, G.Sparano and A.M. Vinogradov, *J. Geom. Phys.*, **14**, 34 (1994).
- [9] Yu.P. Pyt'ev, *Dokl. AN SSSR*, **149**, 298 (1963);
Yu.P. Pyt'ev, *Vestnik Mosk. Univ., Fiz. Astron.*, **2(102)**, 53 (1965) (in Russian).
- [10] V.I. Arnold, "Mathematical Methods of Classical Mechanics". – Springer-Verlag, Berlin, NY, 1980.
- [11] V.I. Arnold, "Singularities of Caustics and Wave Fronts". – Kluwer, Dordrecht, 1990.
- [12] S.N. Kruzhkov, *Matemat. Sbornik*, **98(140)**, 450 (1975) (in Russian).
- [13] V.P. Maslov, "Complex WKB-Method in Nonlinear Equations". – "Nauka", Moscow, 1977 (in Russian).
- [14] R.P. Kerr and W.B. Wilson, *Gen. Rel. Grav.*, **10**, 273 (1979).
- [15] V.V. Kassandrov, "The Algebraic Structure of Space-Time and the Algebrodynamics". – People's Friend. Univ. Press, Moscow, 1992 (in Russian).
- [16] V.V. Kassandrov, *Gravitation & Cosmology*, **3**, 216 (1995); (*e-print* gr-qc / 0007027).
- [17] V.V. Kassandrov, *Acta. Applic. Math.*, **50**, 197 (1998);
V.V. Kassandrov, in: "Quasigroups and Nonassociative Algebras in Physics", eds. J. Löhmus and P. Kuusk. – Inst. Phys. Estonia Press, 1990, p. 202.
- [18] G.C. Debney, R.P. Kerr and A. Schild, *J. Math. Phys.*, **10**, 1842 (1969).
- [19] B. Carter, *Phys. Rev.*, **174**, 1559 (1968).
- [20] C.A. Lopez, *Phys. Rev.*, **D30**, 313 (1984).
- [21] A.Ya. Burinskii, in: "Problems of Theory of Gravitations and Elementary Particles". No.11, ed. K.P. Stan'yukovich. – Atomizdat, Moscow, 1980, p. 47 (in Russian).
- [22] C. Adam, "Hopf maps as static solutions of the complex eikonal equation". – *E-print* math-ph / 0312031.
- [23] V.V. Kassandrov and J.A. Rizcallah, in: "Recent Problems in Field Theory", ed. A.V. Aminova, Kasan Univ. Press, Kasan, 1998, p. 176; (*e-print* gr-qc / 9809078).
- [24] V.V. Kassandrov and J.A. Rizcallah, in: "Fundamental Problems of High Energy Physics and Field Theory", ed. V.A. Petrov. – Inst. High Energy Phys., Protvino, 2002, p. 199; (*e-print* gr-qc / 9809056).
- [25] V.V. Kassandrov and V.N. Trishin, *Gravitation & Cosmology*, **5**, 272 (1999); (*e-print* gr-qc / 0007026).
- [26] V.V. Kassandrov and J.A. Rizcallah, "Twistor and "Weak" Gauge Structures in the Framework of Quaternionic Analysis". – *E-print* gr-qc / 0012109.
- [27] V.V. Kassandrov, *Vestnik People's Friend. Univ., Fizika*, **8(1)**, 34 (2000) (in Russian).
- [28] V.V. Kassandrov, "Singular Sources of Maxwell Fields with "Self-Quantized" Electric Charges". – *E-print* physics / 0308045.
- [29] V.V. Kassandrov, *Gravitation & Cosmology*, **8**, Suppl. 2, 57 (2002); (*e-print* math-ph / 0311006).
- [30] *Instantaneous Action-at-a-Distance in Modern Physics. "Pro" and "Contra"*, eds. A. Chubykalo, V. Pope and R. Smirnov-Rueda. – Nova Science Publ. Inc., NY, 1999.
- [31] Yu.S. Vladimirov and A.Yu. Tur'ygin, "Theory of Direct Particle Interactions". – "Energoatomizdat", Moscow, 1986.
- [32] A.A. Kirillov, *Phys. Lett.*, **B555**, 13 (2003);
A.A. Kirillov and D. Turaev, *Phys. Lett.*, **B532**, 185 (2002).
- [33] V.V. Lychagin, *Acta Applic. Math.*, **3**, 135 (1985).
- [34] L.S. Shikhobalov, *Vestnik S.-Petersb. Univ.*, **1(3)**, 109 (1997); **1(4)**, 118 (1999) (in Russian).

- [35] I.A. Urusovskii, *Zarubezhnaya radioelektronika*, **3**, 3 (1996); **6**, 64 (1996); **6**, 66 (2000).
- [36] M. Saniga, in: *The Nature of Time: Geometry, Physics & Perception*, eds. R. Buccheri, M. Saniga and W.M. Stuckey (NATO Science Series II). – Kluwer Acad. Publ., Dordrecht, 2003, p. 131.
- [37] V.V. Smol'yaninov, *Uspechi Fiz. Nauk*, **170**, 1064 (2000) (in Russian).
- [38] I.A. Shelaev, “Introduction to irreversible electrodynamics”. – Dubna, 1999 (in Russian).
- [39] N.A. Koz'yrev, “Selected Papers”. – Leningrad, 1991;
N.A. Koz'yrev, in: *History and Methodology of Natural Sciences. No.2*. – Inst. Phil. Press, Moscow, 1963, p. 95 (in Russian).
- [40] A.P. Levich, in: *Construction of Time in Natural Sciences: On the Way to Understanding of Time Phenomenon*, ed. A.P. Levich. – Moscow Univ. Press, Moscow, 1996 (in Russian); A.P. Levich, *Gravitation & Cosmology*, **1(3)**, 237 (1995).
- [41] V.V. Kassandrov, in: *Mathematics and Practice. Mathematics and Culture. No. 2*, ed. M.Yu. Simakov. – “Self-Education”, Moscow, 2001, p. 67 (in Russian).