On Mathematical Foundations and Physical Applications of Chronometry

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1. Motivation and Introduction

By way of introduction I begin with justifying the statement *The chronometric theory by* I.Segal *is the crowning accomplishment of special relativity* which I made in the title of my earlier survey article [Le93].

The term "world" which we shall use below is close to the term "space-time" [SaWu, p.27]; however, it does not assume that we fix a particular Lorentzian metric tensor field from the conformal class [Se76, GuSt]. Since the present article is dedicated mostly to the conformal compactification $\overline{\mathbf{M}_0}$ of the Minkowski world $\mathbf{M}_0$ and its universal coverings, I shall not go into the general definitions specifying space-times and causality.

I begin with *Newtonian world* $\mathbf{N}$, namely, a 4-dimensional affine space equipped with a "Newtonian causal structure" [Se76, p.23]. The latter is defined as the family $\{\mathbf{J}_x^+ : x \in \mathbf{N}\}$ of closed half-spaces with parallel boundary hyperplanes. An "event" x belongs to its "future set" $\mathbf{J}_x^+$. The symmetry group $\mathbf{S}$ is the 11-dimensional Galilean group (including scaling) [GuSt]. The group $\mathbf{S}$ yields the Euclidean geometry of absolute 3-space.

Next, *Minkowski world* $\mathbf{M}_0$ is defined as 4-dimensional affine space, but its causal structure $\{\mathbf{J}_x^+ : x \in \mathbf{M}_0\}$ consists of elliptic convex cones such that $\mathbf{J}_y^+$ is obtained from $\mathbf{J}_x^+$ by parallel translation $z \mapsto z + y - x$. The symmetry group $\mathbf{P}$ is the 11-dimensional Poincaré group (included scaling). It was H. Minkowski who insisted on the "absolute" status of space-time (instead of that of space). He also raised the question of transforming the structures involved into less degenerate ones (for his "anti-deformation" thesis, see e.g. the Introduction of [ØrSe]). The well-known result by A. D. Alexandrov (see e.g. [Gu] for exact references on the subject of this and other matters discussed in this section), later rediscovered in a weaker version by E. C. Zeeman, originated the method of deriving the geometry from the causal structure. The usage of $\mathbf{P}$ instead of the "standard" 10-dimensional $\mathbf{P}_0$ is motivated by this very method when applied to $\mathbf{M}_0$. It is worth noting that matters are quite different with the symmetry group of the Newtonian world: there are many transformations of **N** which preserve the causal structure even though they do not belong to **S**.

Finally we discuss briefly the main aspects of chronometry. Its world **M** consists of the Einstein space-time **E** as the underlying conformal manifold. The metric on **E** is $dt^2 - ds^2$, where t is time and ds^2 is the Euclidean on $\mathbb{S}^3$ induced by the the standard immersion of $\mathbb{S}^3$ into $\mathbb{R}^4$. A future time direction in **E** is fixed, and a "future cone" appears in every tangent space of **M**. In **M** one can define "future sets" [Se76] in a fashion similar to the previous cases. This structure gives rise to the symmetry group $\widetilde{\mathbf{G}}$ which is now the universal covering of $SU(2,2)$. It acts globally on **M**. These and other notions will be defined in the next section in a greater detail. The Minkowski world is conformally imbedded into **M** via the "Cayley transform". The radius R of the space $\mathbb{S}^3$ does not depend on the chosen metric from this conformal class, i.e., the metric in which it is calculated. In other words, R is a conformal invariant [Se82]. It is convenient to use natural *chronometric units* in which R, the speed of light c, and the Plank constant $\hbar$ are equal to 1. We denote by $\widetilde{\mathbf{K}}$ the 7-dimensional Einstein isometry group. A subgroup of a Lie group is said to be *essentially compact* if its image under the adjoint representation is compact. Now $\widetilde{\mathbf{K}}$ is a maximal essentially compact subgroup of $\widetilde{\mathbf{G}}$ It consists of translations in time and rotations in space $(\rightarrow 2.2)$[1].

Several features that important in applications; I shall indicate them without defining explicitly the mathematical definitions of the notions involved. I shall reproduce a small piece from [Se91] almost without changes.

The chronometric energy $\mathcal{H}$ is the generator of time in **E**. Relative to any point of observation in **M**, the Minkowski world $\mathbf{M}_0$ is imbedded **P**-covariantly, and the relativistic or Minkowski energy $calH_0$ is the generator of time in $\mathbf{M}_0$ relative to the Lorentz frame in $\mathbf{M}_0$, which, at the point of observation, osculates the frame defined by the space-time splitting in **E**. For any unitary positive-energy representation of $\widetilde{\mathbf{G}}$, the corresponding Einstein energy exceeds the Minkowski energy by an amount that vanishes infinitesimally but increases with the spatial support of the state in question in terms of the appropriate quantum mechanical consideration. The inertial mass of a cosmologically long-lived particle is represented in accordance with Mach's principle as its interaction energy with the cosmic backround and is correspondingly only $\widetilde{\mathbf{K}}$-invariant, implying approximate *local* $\mathbf{P}_0$-invariance of its rest mass.

Additional background on chronometry is given in SEGAL's book [Se76] and [PS-I,II; P-III, P-IV, ØrSe, Se86]. In these articles the physical particles have been modelled, in accordance with the thrust of decades of theoretical investigation in this area, by induced bundles $(\rightarrow 4.1)$ over causally oriented space-times.

Let me now conclude with the justification of the expression " crowning accomplishment of special relativity". Firstly, the conformal group is semisimple, in constrast with the Poincaré group . Hence it cannot be regarded as resulting through a contraction process from a non-isomorphic Lie group of the same di-

1 I shall try to make the presentation as self-contained as possible. Cross-references in the text are indicated by arrows $\rightarrow$.

mension. Secondly, it arises as maximal *local* causal group of the special relativistic world in which only the 11-dimensional Poincaré group can be globally realized. When compared with other theories based on the world of special relativity or particular space-times of general relativity, Chronometry has other preferable features; we mention only a few:

—the absence of the fixed Lorentzian structure which seems to be connected with a concrete *metric observer* [Se76] in the world under consideration,

—a better unification of elementary particles ($\rightarrow 6.1 - 6.3$),

—the existence of *leaking* $(6.1, 6.3)$ which gives kinematic explanation of several *decays* (6.2, 6.3).

In discussing chronogeometry it is worthwhile to mention that there are exactly four 4-dimensional real Lie algebras which admit an invariant nondegenerate form of Lorentzian signature [GuLe, Le86′]. Such a form is a well-known to correspond to a biinvariant metric on the Lie group in question. The above and several other facts (see [Se76, Se86, GuLe, Le86, Le86′, Le93] and references therein) support the conclusion that **M** is the “basic world” of Nature and that it is, together with the Minkowski space-time, one of the most important ones in the applications.

Summing up we note that chronometry is derived from very general considerations of causality, stability, and symmetry. Therefore, it is somewhat abstract, and its empirical implications call for further development. Indeed I consider it as one of the my goals in the present survey to convince the specialists in relativity that Chronometry is an effective point of departure for cosmology and that they should take part in its implementation and further development. Chronometry, like special relativity and quantum mechanics, may initially appear contradictory to accepted doctrine. But its application to extragalactic astronomy ($\rightarrow$ 7, [SeNi] and references therein) has shown that it is capable of precise and detailed predictions regarding the *cosmic redshift* ($\rightarrow$ 7) and other directly measured quantities, in spite of its lack of adjustable cosmological parameters.

Remark. I use the opportunity to mention that the proof of Lemma 2 in [Le86′] should be slightly modified. Notably, the subspace S occuring in this article need not be a subalgebra. I am indebted to A. Kuzemchikov who has pointed this out to me.

2. Synthetic Geometry

2.1. The “Hermitian” model of the Minkowski world

Fix the following representation of Pauli matrices:

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The set of all 2 by 2 Hermitian matrices is denoted by $\mathbf{H}$; each X from $\mathbf{H}$ has a unique decomposition

$$X = x^m \sigma_m. \tag{$*$}$$

Here and henceforth we assume the Einstein summation rule. (We allow also the use of lower indices for coordinates of the vectors under consideration). Let an orthonormal coordinate system with basis e_0, e_1, e_2, e_3 be chosen in the Minkowski world $\mathbf{M}_0$. The map from $\mathbf{M}_0$ to $\mathbf{H}$ which takes $x = x^m e_m$ into X via equation $(*)$ is a linear bijection. Then $\det X = {x_0}^2 - {x_1}^2 - {x_2}^2 - {x_3}^2$. Let the future cone in $\mathbf{M}_0$ of an event x be denoted by J_x^+. Then the causal relation $y \in J_x^+$ in $\mathbf{M}_0$ holds if and only if the matrix $Y - X$ is positive semidefinite [Se76]. The restricted 10-dimensional Poincaré group $\mathbf{P}_0$ is the semi-direct product $\mathbf{P}_0 = \mathbf{H} \rtimes_{\Lambda_0}$ of the vector group $\mathbf{H}$ and the restricted 6-dimensional Lorentz group $\mathbf{\Lambda}_0$. Its universal covering $\widetilde{\mathbf{P}}_0$ equals $\mathbf{H} \rtimes_S L(2, \mathbb{C})$, where $(F, L) \in \mathbf{H} \rtimes SL(2, \mathbb{C})$ acts in $\mathbf{H}$ by $H \mapsto LH\bar{L}^T + F$.

The simply connected Poincaré group including scaling is denoted by $\widetilde{\mathbf{P}}$. It is the semi-direct product of $\mathbf{H}$ by the group $\widetilde{\Lambda} = R^1 \times SL(2, \mathbb{C})$, and the element $(F, (t, L)) \in \widetilde{\mathbf{P}}$ acts on $\mathbf{H}$ as $H \mapsto e^t LH\bar{L}^T + F$ with $t \in \mathbb{R}^1$ and $(L, F) \in \widetilde{\mathbf{P}}_0$. The conventionally defined 7-dimensional Lorentz group including scaling is denoted by Λ.

It is a well known fact that $\Lambda_0 \cong SL(2, \mathbb{C})/\{\mathbf{1}, -\mathbf{1}\}$ Accordingly, the groups $\widetilde{\Lambda}$, $\widetilde{\mathbf{P}}$, and $\widetilde{\mathbf{P}}_0$ doubly cover Λ, $\mathbf{P}$, and $\mathbf{P}_0$), respectively.

Next we fix the Hermitian form $\langle \cdot, \cdot \rangle$ in $\mathbb{C}^2$:

$$\langle x, x' \rangle = x_1 \bar{x}_1' + x_2 \bar{x}_2'. \tag{1}$$

The group $\mathbf{U(2)}$ is the totality of those linear transformations of $\mathbb{C}^2$ which preserve (1). The group $\mathbf{U(2)}$ is generated by exponentials of matrices iF where $F \in \mathbf{H}$; that is, the Lie algebra of $\mathbf{U(2)}$ is exactly $i\mathbf{H}$.

In the complex linear space $\mathbb{C}^4$ with a fixed decomposition as $\mathbb{C}^2 \oplus \mathbb{C}^2$ we introduce the form

$$\langle\langle x \oplus y, x' \oplus y' \rangle\rangle = \langle x, x' \rangle - \langle y, y' \rangle. \tag{2}$$

Then, by definition, $SU(2,2)$ is the totality of those linear unimodular transformations of $\mathbb{C}^4$ which preserve (2). We abbreviate $SU(2,2)$ by $\mathbf{G}$. The elements of $\mathbf{G}$ are written as suitable block matrices

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}, \qquad A,\ B,\ C\ D \in M_2(\mathbb{C})$$

The following ("linear-fractional") action of $\mathbf{G}$ on $\mathbf{U(2)}$ plays the crucial role in the further development of the theory:

$$gZ = (AZ + B)(CZ + D)^{-1}, \qquad g = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \quad Z \in U(2). \tag{3}$$

There exists a bi-invariant field of (tangential) elliptic cones on $\mathbf{U(2)}$ [Se76] depending on one real parameter [GuLe, Le86′] which may be interpreted as the speed of light c. If units are chose so that $c = 1$, then the cone at $\mathbf{1} \in \mathbf{U}(2)$ becomes essentially the one already fixed in $\mathbf{H}$ (up to multiplication by i when passing from $\mathbf{H}$ to the appropriate Lie algebra) .

Segal's world $\mathbf{M}$ which has already been mentioned in the introduction *is defined* as the universal cover of $\mathbf{U(2)}$. The causal structure of $\mathbf{M}$ thus arises naturally. Recall that the static Einstein universe $\mathbf{E}$ is one of the most significant space-times of General Relativity. Its underlying topological space $\mathbb{R}^1 \times \mathbb{S}^3$ is the same as that of $\mathbf{M}$. The prescribed Lorentzian metric of $\mathbf{E}$ equals $dt^2 - ds^2$, where $t \in \mathbb{R}^1$, and ds is the element of arc length on $\mathbb{S}^3$. This Lorentzian metric defines the same cone field as already introduced on $\mathbf{M}$.

The action (3) on $\mathbf{U}(2)$ is canonically pulled back to a $\widetilde{\mathbf{G}}$-action on $\mathbf{M}$. The fundamental characteristic of the latter is that it preserves the "infinitesimal" causal structure (consisting of tangential future cones) as well as the "global" one (consisting of the future sets in $\mathbf{M}$ itself). Alternatively, the group of transformations satisfying these condition (the *causal* group) is the group of all conformal transformations [Se76, Le87 and references therein].

We use the following notation:

$$g^{-1} = \begin{pmatrix} A' & B' \\ C' & D' \end{pmatrix},$$

$$\mathbf{\Omega} = 2^{-1/2} \begin{pmatrix} \mathbf{1} & \mathbf{1} \\ -\mathbf{1} & \mathbf{1} \end{pmatrix}.$$

$$\Omega^{-1} g^{-1} \Omega = \begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix},$$

$$\Omega^{-1} g \Omega = \begin{pmatrix} A'' & B'' \\ C'' & D'' \end{pmatrix}. \tag{4}$$

Exercise 1. The following relations hold:

$$A'' = (1/2)(A - B - C + D), \quad A = (1/2)(A'' + B'' + C'' + D''),$$

$$B'' = (1/2)(A + B - C - D), \quad B = (1/2)(-A'' + B'' - C'' + D''),$$

$$C'' = (1/2)(A - B + C - D), \quad C = (1/2)(-A'' - B'' + C'' + D''),$$

$$D'' = (1/2)(A + B + C + D), \quad D = (1/2)(A'' - B'' - C'' + D'').$$

$$A' = A^*, \quad A'' = D_1^*,$$

$$B' = -C^*, \quad B'' = B_1^*,$$

$$C' = -B^*, \quad C'' = C_1^*,$$

$$D' = D^*, \quad D'' = A_1^*.$$

□

Exercise 2. The following conditions are equivalent for an element $g \in SL(\mathbb{C}^2 \oplus \mathbb{C}^2)$ of the form (3). Each of the two is necessary and sufficient for g to lie in $SU(2,2)$:

(i)
$$A^*A - C^*C = \mathbf{1}, \quad B^*B - D^*D = -\mathbf{1},$$
$$A^*B - C^*D = \mathbf{0}.$$

(ii) For an element of the form (4) the following relations hold:
$$A''^*D'' + C''^*B'' = \mathbf{1}, \quad A''^*C'' + C''^*A'' = \mathbf{0},$$
$$B''^*D'' + D''^*B'' = \mathbf{0}. \qquad \square$$

2.2. Imbeddings of $\mathbf{M_0}$ into $\bar{\mathbf{M}}_\mathbf{0}$ and $\mathbf{M}$

These two imbeddings are defined and considered as canonical ones. As a consequence, the causal group $\widetilde{\mathbf{P}}$ of $\mathbf{M}_0$ is carried into a subgroup of the causal group $\widetilde{\mathbf{G}}$ of the world $\mathbf{M}$. Interestingly, the isotropy group in the action on $\mathbf{M}$ of $\widetilde{\mathbf{G}}$ is the causal group of $\mathbf{M_0}$.

Theorem 2.1 [PS-I]. *$\widetilde{\mathbf{G}}$ acts causally and transitively on $\mathbf{M}$, with isotropy group isomorphic to $\widetilde{\mathbf{P}}$.* $\square$

The proof in [PS-I] proceeds via several lemmas which we provide with minor modifications and without proofs. The isomorphism from $\widetilde{\mathbf{P}}$ to the isotropy subgroup of the point $p = (\pi, \mathbf{1})$ in $\mathbf{M}$ is denoted by χ and will be used later. Depending on the context, $\overline{\mathbf{M}_0}$ stands for the unitary group $\mathbf{U}(2)$.

Lemma 1. *The Cayley map $\mathbf{c} : \mathbf{M}_0 \to \overline{\mathbf{M}_0}$, defined by the equation*
$$\mathbf{c}F = (\mathbf{1} + iF/2)(\mathbf{1} - iF/2)^{-1}, \quad F \in \mathbf{H},$$
is causal. Its image is dense and open in $\overline{\mathbf{M}_0}$. $\square$

The $\widetilde{\mathbf{G}}$-action on $\mathbf{M}$ is defined canonically in pull-up terms: if $\tilde{g} \in \widetilde{\mathbf{G}}$, $\tilde{z} \in \mathbf{M}$, and if $\sigma : M \to U(2)$ and $\gamma : \widetilde{G} \to G$ are the converings, then an element $\tilde{g}\tilde{z} \in M$ is defined uniquely by the condition
$$\sigma(\tilde{g}\tilde{z}) = \gamma(\tilde{g})\sigma(\tilde{z}).$$
From the introduction we know that the underlying topological space of $\mathbf{M}$ is $\mathbb{R}^1 \times \mathbb{S}^3$. More concretely, if $\mathbf{M}$ is identified with $\mathbb{R} \times SU(2) \cong \mathbb{R}^1 \times \mathbb{S}^3$, then
$$\sigma(t, W) = e^{it}W, \quad t \in \mathbb{R}, \ W \in SU(2).$$
The covering γ will be described later.

The terminology *maximal essentially compact connected* subgroup of $\mathbf{G}$ was introduced in the Introduction; we shall briefly speak of MECC subgroups. All of them are conjugate by some element of $\mathbf{G}$.

Lemma 2. *A MECC subgroup* $\widetilde{\mathbf{K}}$ *of* $\widetilde{\mathbf{G}}$ *is isomorphic to*

$$\mathbb{R}^1 \times SU(2) \times SU(2).$$

Normalizing $\widetilde{\mathbf{K}}$ *thus and* σ *as above, the following equations hold for* $\tilde{k} = (s, U, V)$ *and* $\tilde{z} = (t, W)$*:*

$$\tilde{k}\tilde{z} = (s+t) \times UWV^{-1}, \qquad \gamma(k) = \begin{pmatrix} e^{is/2}U & 0 \\ 0 & e^{-is/2}V \end{pmatrix}. \qquad \square$$

Lemma 3. *There exists a unique isomorphism* β *of* $\widetilde{\mathbf{P}}$ *into* $\mathbf{G}$ *such that the equality* $\beta(g)\mathbf{c}(H) = \mathbf{c}(gH)$ *holds for all* $g \in \widetilde{\mathbf{P}}$ *and* $H \in \mathbf{H}$. $\square$

The constructive proof in [PS-I] produces for (t, L, F) from $\widetilde{\mathbf{P}}$, in particular,

$$\beta(T) = \Omega \begin{pmatrix} e^{t/2}L & (i/2)e^{-t/2}F(L^{*-1}) \\ 0 & e^{-t/2}(L^{*-1}) \end{pmatrix} \Omega^{-1}.$$

Lemma 4. $\beta(\widetilde{\mathbf{P}})$ *is the component of the identity in* $\mathbf{G}_{-1}$. $\square$

Lemma 5. *Let* $\mathbf{Z}$ *denote the center of* $\mathbf{G}$*. Then* $\mathbf{G}_{-1} = \beta(\widetilde{\mathbf{P}})\mathbf{Z}$. $\square$

Several statements follow from Theorem 2.1 which are of sufficient importance to be reproduced here.

Corollary 2.1.1. *The actions of* $\widetilde{\mathbf{P}}$ *on* $\mathbf{H}$ *and of the isotropy subgroup of* $\widetilde{\mathbf{G}}$ *on an open orbit in* $\mathbf{M}$ *define identical causal transformation groups:* $\mathbf{H}$ *is causally equivalent to the orbit via a map intertwining the respective actions of* $\widetilde{\mathbf{P}}$. $\square$

Corollary 2.1.2. *The center of* $\widetilde{\mathbf{G}}$ *is generated by the two elements of* $\widetilde{\mathbf{K}}$*:*

$$\zeta = (\pi, e^{-i\pi}, \mathbf{1}), \quad \eta = (0, e^{i\pi}, e^{i\pi}). \qquad \square$$

Remark 1. It follows that the center of $\widetilde{\mathbf{G}}$ is of the form $\mathbf{Z}_\infty \times \mathbf{Z}_2$, where the $\mathbf{Z}_2$ component acts trivially on $\widetilde{\mathbf{U}}(2)$.

Corollary 2.1.3. *Two points of* $\mathbf{M}$ *are left fixed by the same isotropy subgroup of* $\widetilde{\mathbf{G}}$ *if and only if one is the transform of the other by an element of the center of* $\widetilde{\mathbf{G}}$. $\square$

2.3. Discrete Causal Symmetries

Any transformation of M_0 reverting the causal structure is called *anticausal.* The "time reversal" $T_0\colon (x^0,\vec{x}) \to (-x^0,\vec{x})$ in $\mathbf{M}_0$ is a typical example of an anticausal transformation. If Minkowski world is represented by $\mathbf{H}$, then the group of all causal and anticausal transformations on is denoted by $\mathbf{G}^+(\mathbf{H})$. It is generated by the connected component $\mathbf{G}_0^+(\mathbf{H})$ of the identity together with T_0 and space reversal $P_0\colon (x^0,\vec{x}) \mapsto (x^0,-\vec{x})$. Note $\mathbf{G}^+(\mathbf{H})/\mathbf{G}_0^+(\mathbf{H}) \cong \mathbf{Z_2} \times \mathbf{Z_2}$.

$\mathbf{P}^+$ (respectively, $\mathbf{G}^+$) denotes the group generated by $\mathbf{P}$ (respectively, $\mathbf{G}$) and these discrete symmetries.

We now define $\widetilde{\mathbf{P}}^+$ and $\widetilde{\mathbf{G}}^+$: By a *basic pair of discrete symmetries* (P,T) at a point H of $\mathbf{H}$ we mean an ordered pair of causal, respectively, anticausal transformations on $\mathbf{H}$ of the form:

$$P = S^{-1}P_0S, \quad T = S^{-1}T_0S,$$

where S is a transformation in $\mathbf{P}$ that carries H into 0 and P_0 and T_0 are the transformations on $\mathbf{H}$:

$$P_0 : H \mapsto trH - H, \quad T_0 : H \mapsto H - trH.$$

Since P_0 and T_0 are causal and anticausal, respectively, the same must be true of P and T. Note that $P^2 = \mathbf{1} = \mathbf{T^2}, \mathbf{PT} = \mathbf{TP}$. It follows that a basic pair of dicrete symmetries at a given point is unique within conjugation by a causal transformation connected to the identity that leaves the point fixed.

P and T generate $\mathbf{F} \simeq \mathbf{Z_2} \times \mathbf{Z_2}$. The semi-direct product of $\mathbf{P}$ with $\mathbf{F}$ gives $\mathbf{P}^+$. The action of $\mathbf{F}$ as a group of automorphisms in $\mathbf{P}$ is canonically extended to an action on $\widetilde{\mathbf{P}}$ and the semi-direct product of $\mathbf{F}$ with $\widetilde{\mathbf{P}}$ relative to this action forms the universal cover $\widetilde{\mathbf{P}}^+$. The natural projection of $\widetilde{\mathbf{P}}^+$ onto $\mathbf{P}^+$ is independent of the choice of a basic pair and of a base point H.

The notion of a basic pair of discrete symmetries at a point is now canonically extended to $\mathbf{U}(2)$. Introduce the following transformations of $\mathbf{U}(2)$:

$$P_0 : U \mapsto (\det U)U^{-1}, \quad T_0 : U \mapsto (\det U)^{-1}U.$$

A basic pair of discrete symmetries at a point V of $\mathbf{U}(2)$ is defined as an ordered pair of

$$P = S^{-1}P_0S, \quad T = S^{-1}T_0S,$$

where S is a transformation in $\mathbf{G}$ that carries V into $\mathbf{1}$. Such a basic pair intertwines with one on $\mathbf{H}$, when V is in the range of the caley map ($\to$ 2.2). It follows from this that the basic pair of discrete symmetries at a point of $\mathbf{U}(2)$ is unique within conjugation by an element of $\mathbf{G}$ leaving the point fixed. One gets $\widetilde{\mathbf{G}}^+$ as the semi-direct product of $\mathbf{F}$ with $\widetilde{\mathbf{G}}$, where the action of $\mathbf{F}$ on $\widetilde{\mathbf{G}}$ is defined by a canonical extension to $\widetilde{\mathbf{G}}$ of the action of $\mathbf{F}$ on $\mathbf{G}$ just defined.

Corollary 2.1.4. *$\widetilde{\mathbf{G}}^+$ acts on $\mathbf{M}$ causally or anticausally, extending the action of $\widetilde{\mathbf{G}}$ given in* Theorem 2.1*, and with isotropy subgroup $\widetilde{\mathbf{P}}^+$. Moreover, there is a causal equivalence between an open orbit of this isotropy group and $\mathbf{H}$ that intertwines the respective actions of $\widetilde{\mathbf{P}}^+$ on them, canonically identifiable with that given in the theorem.* □

In the proof in [PS-I] the β ($\to$ 2.2) is extented to $\widetilde{\mathbf{P}}^+$ with the conservation of the intertwining relation of the theorem. The χ of the theorem extends similarly.

Note that Corollary 2.1.2 does not extend. The center of $\widetilde{\mathbf{G}}^+$ is generated by ζ^2 and η. Due to its failure to commute with P, the element ζ is no longer in the center. In terms of the connected group, the result may be stated as follows.

Corollary 2.1.5. *The elements ζ and η of the center of $\widetilde{\mathbf{G}}$ are both invariant under T, and η is invariant under P, but $P^{-1}\zeta P = \zeta\eta$.* □

Remark 2. Thus every causal or anti-causal transformation on $\mathbf{M}_0$ corresponds to a unique such transformation on $\mathbf{M}$ that agrees on $\mathbf{M}_0$ regarded as imbedded in $\mathbf{M}$ causally, with the given transformation.

2.4. More on Chronometric Geometry

The map $(e^{it}, V) \mapsto e^{it}V : U(1) \times SU(2) \to U(2)$ is a double covering. Denote the domain manifold by $\mathbf{M}^{(2)}$. It is equipped with an (infinitesimal) causal structure and the parametrization $(u_{-1}, u_0) \times (u_1, u_2, u_3, u_4)$ subject to the condition that $u_{-1}^2 + u_0^2 = u_1^2 + \cdots + u_4^2 = 1$. Here is the presentation of a general element in $U(2)$:

$$(u_{-1} + iu_0)(iu_1\sigma_1 + iu_2\sigma_2 + iu_3\sigma_3 + u_4).$$

When $\mathbf{M}_0$ is embedded into $\mathbf{U}(2)$ via the Cayley map then the coordinates x_0, x_1, x_2, x_3 in $\mathbf{M}_0$ agree with the $u_m (m = 0, 1, 2, 3)$ within terms of second order in the x_m, ($\to$ 3.3).

Note that although from a Minkowskian standpoint, $(\pi, \mathbf{1})$ appears infinitely distant in time and $(0, -\mathbf{1})$ appeares infinitely distant in space, from the point of observation $(0, \mathbf{1})$ which corresponds to the origin in $\mathbf{M}_0$, they both cover the same point $-\mathbf{1}$ of $\mathbf{U}(2)$; the space-time separation in $\mathbf{M}$ is only infinitesimally the same as that in $\mathbf{M}_0$.

Four presentations of the infinitesimal causal symmetries are given in Table I of [SeJa]. It is presupposed there that the real projective quadric

$$q_{-1}^2 + q_0^2 - q_1^2 - q_2^2 - q_3^2 - q_4^2 = 0,$$

where the q_m parametrize a point in projective 5-space) has a unique causal structure invariant under the group of projectivities that leave it fixed (within reversal) which is locally $SO(2,4)$ and is equivalent to $\mathbf{U}(2)$ [Se76]. The vector fields on $\mathbf{U}(2)$ corresponding to the operators

$$\epsilon_s q_s \partial_m - \epsilon_m q_m \partial_s$$

are consequently infinitesimal causal symmetries and, when lifted to $\mathbf{M}^{(2)}$, are denoted as L_{sm}, forming the entries of Table I, column 1 (here and further on, the row ϵ of the six elements ϵ_m equals $(1,1,-1,-1,-1,-1)$). Column 2 gives the expressions for the L_{sm} as linear combinations of the X_m where the latter are the generators of one-parameter groups of transformations lifted from the action $U \mapsto U\exp(it\sigma_m)$ on $\mathbf{U}(2)$. Expressions for the X_m in terms of the L_{sk} are:

$$\begin{aligned} X_0 &= L_{-10}, \\ X_1 &= L_{14} - L_{23}, \\ X_2 &= L_{24} - L_{31}, \\ X_3 &= L_{34} - L_{12}. \end{aligned}$$

In the third column of Table I the L_{sm} are *antirepresented* as concrete matrices in $su(2,2)$, with commutation relations (3.1) below. The fourth (and the last) column of Table I expresses L_{sm} as vector fields in $\mathbf{M}_0$. The flat limit of the L_{sm} is seen from Column 4 by replacing x_m by x_m/R in which R is the "radius of the universe" $\mathbb{S}^3$ in laboratory units, then rescaling L_{sm} appropriately, and finally forming the limit as $R\to\infty$.

3. Analytic Geometry

3.1. Preliminaries

It is convenient to make an explicit distinction between a generator of the group $\mathbf{G}$ and the corresponding vector field on $\mathbf{M}$. Generators of the abstract group will be denoted by bfface letters; corresponding vector fields on $\mathbf{M}$ by the same Roman capital letter. Note that the mapping $\mathbf{X} \mapsto X$ from $\mathcal{G}$ to the space of vector fields on $\mathbf{M}$ is an anti-representation of the Lie algebra $\mathcal{G}$. Thus, in terms of the $SO(2,4)$-generators $\mathbf{L}_{im}$, the commutation relations

$$[\mathbf{L}_{im}, \mathbf{L}_{mk}] = -\epsilon_m \mathbf{L}_{ik} \tag{1}$$

are opposite in sign compared with the ones between the corresponding vector fields L_{im}:

$$[L_{im}, L_{mk}] = \epsilon_m L_{ik}.$$

If the $\mathbf{L}_{sm}$ are regarded as elements of $su(2,2)$ as in part of Table 1 of [SeJa], they satisfy the set (1) of commutation relations.

3.2. Infinitesimal Causal Symmetries in Polar Coordinates

The Table I is supplemented by presentation of vector fields L_{sm} in polar coordinates (which are convenient when regarding $\mathbf{M}$) in Table II of [PS-I]. These coordinates have the following relations to the u_m:

$$e^{it} = u_{-1} + iu_0, \quad u_1 = \sin\rho\sin\theta\cos\phi, \quad u_2 = \sin\rho\sin\theta\sin\phi,$$

$$u_3 = \sin\rho\cos\theta, \quad u_4 = \cos\rho;$$

here $0 \leq \theta$, $\phi \leq \pi$, $0 \leq \phi \leq 2\pi$.

3.3. Relations between the x_m and the u_m

The *standard imbedding* of $\mathbf{M}_0$ into $\mathbf{M}$ takes an (x_0, x_1, x_2, x_3) to (t, U), where

$$u_{-1} = p(1 - x^2/4), u_j = px_j, u_4 = p(1 + x^2/4),$$

$$p = \left((1 - x^2/4)^2 + {x_0}^2\right)^{-1/2}, e^{it} = u_{-1} + iu_0,$$

$$U = u_4 + u_1b_1 + u_2b_2 + u_3b_3, \quad -\pi < t < \pi,$$

and b_m stands for $i\sigma_m$, ($\rightarrow$ 2.1). This mapping, followed by the covering map of $\mathbf{M}$ onto $\mathbf{M}^{(2)}$ (whose coordinates are the u_m) is one-to-one. A point of $\mathbf{M}^{(2)}$ corresponds to a point of $\mathbf{M}_0$ if and only if $u_{-1}+u_4 > 0$; the Minkowski coordinates are recovered by the equation $x_m = 2u_m(u_{-1} + u_4)^{-1}$.

The function p is strictly positive on $\mathbf{M}_0$ and is extended smoothly to $\mathbf{M}^{(2)}$ by $p = (u_{-1} + u_4)/2$, and then to $\mathbf{M}$ via polar coordinates ($\rightarrow$ 3.2).

3.4. Expressions for the Right- and Left-Invariant Symmetries

The above introduced vector fields X_m are left-invariant on $\widetilde{\mathbf{U}}(2)$ though generating right translations on it. In Table III of [PS-I] the corresponding generators Y_m of left translations are presented (as well as the X_m) as linear combinations of the L_{sk} and as concrete vector fields in polar coordinates. Their commutation relations are given therein.

3.5. Actions of Relevant Vector Fields on the u_m

Table IV in [PS-I] gives Lu_m for various vector fields L involved.

3.6. Basic Flat and Inverted Generators

Conformal inversion in $\mathbf{M}_0$ is defined as the map $\mathbf{x} \mapsto 4\mathbf{x}/\mathbf{x}^2$, where defined. This map extends uniquely to the everywhere-defined smooth map $U \mapsto -U/\det U$ on

$\mathbf{U}(2)$. A corresponding map on $\mathbf{M}$ is ambiguous within an element of the center of $\widetilde{\mathbf{G}}$. In [PS-I] this element is standardized and *causal inversion* is defined on $\mathbf{M}$ as the map $(t,V) \mapsto (-t,-V)$. Conformal inversion on $\mathbf{M}_0$ carries the ∂_m into vector fields $\hat{\partial}_m$ that are sometimes called "special conformal (infinitesimal) transformations" when extended by continuity to be everywhere defined on $\mathbf{M}_0$. The generators of the Lie algebra of $\widetilde{\mathbf{G}}$ that correspond to the vector fields ∂_m and $\hat{\partial}_m$ on $\mathbf{M}_0$ are called the *basic flat* and *inverted* generators, and denoted as T_m and $\hat{T}_m$, respectively. In these terms the relation $\mathbf{L}_{-1m} = \mathbf{T}_m - \hat{\mathbf{T}}_m \quad (m = 0,1,2,3)$ stands. In Table V of [PS-I] these eight generators are expressed in terms of the $\mathbf{L}_{sm}$ and in terms of the ∂_m.

3.7.Metrics, Measures, Forms

The following objects are introduced as *standard* in the corresponding section of [PS-I]: The *flat metric* on $\mathbf{M}_0$, the *curved metric* on $\mathbf{M}$, the *flat measure* in $\mathbf{M}_0$, the *curved measure* in $\mathbf{M}$, and so on.

3.8. Enveloping Algebra Relations

Let us use the following notations for designated elements of the universal enveloping algebra $\mathcal{E}$ of the Lie algebra $\mathcal{G}$:

$$\mathbf{S} = \mathbf{L}_{-\mathbf{14}}, \quad \mathbf{L_f} = \mathbf{T_0^2} - \mathbf{T_1^2} - \mathbf{T_2^2} - \mathbf{T_3^2},$$

$$\triangle = (\mathbf{X_1^2} + \mathbf{X_2^2} + \mathbf{X_3^2} + \mathbf{Y_1^2} + \mathbf{Y_2^2} + \mathbf{Y_3^2})/\mathbf{2},$$

$$\mathbf{L}_c = \mathbf{X}_0^2 - \triangle.$$

The *curved* or *chronometric* (respectively: *flat* or *relativistic*) *Hamiltonian* H (respectively: H_0) is defined as the image (up to multiplication by i) of $\mathbf{X}_0$ (respectively: of $\mathbf{T}_0$) under the corresponding representation.

The following relation holds in $\mathcal{E}$:

$$4\mathbf{L}_f = [\mathbf{S}, \mathbf{L_c}] + [\mathbf{S}, [\mathbf{S}, \mathbf{L_c}]]/\mathbf{2}.$$

For a given element $\mathbf{Q}$ of $\mathcal{E}$, the corresponding differential operator on $\mathbf{M}$, obtained by extending $\mathbf{X} \mapsto \mathbf{X}$ as an antirepresentation, is denoted by the same (non-bfface) letter.

The following relation holds:

$$[S, L_c] = -2u_{-1}u_4L_c - 2u_0u_4X_0 + 2u_{-1}(u_1X_1 + u_2X_2 + u_3X_3).$$

3.9. Scale Actions.

It is mentioned in [PS-I] how useful the next statement is. It can be proved via using $p = (u_{-1} + u_4)/2$ and Table IV.

Scholium 3.1. *For any real constant* k, $Sp^k = -kp^k(1 - u_{-1}u_4)$. *Moreover, if* $g = e^{tS}$, *then*
$\left(\frac{\partial}{\partial t}\right)\ [d_4(g^{-1}u)/d_4u]_{t=0} = -4u_{-1}u_4,$
and

$$\left(\frac{\partial}{\partial t}\right)\ [d_3(g^{-1}u)/d_3u]_{t=0} = -3u_{-1}u_4.$$

The reader is referred to Section 3.7 for the notions involved. The 4- and 3-forms d_4u, d_3u on $\mathbf{M}$ are introduced in [PS-I].

4. Conformal group actions in induced bundles

4.1. Basic constructions and applications

Let us describe briefly the notion of an *induced* representation. Let G be a Lie group and H a closed subgroup. Set $X = G/H = \{gH : g \in G\}$ and assume that G acts on the left on X. Let $(g, f) \mapsto gf\colon H \times F \to F$ denote a linear action of H on a finite dimensional vector space F. In $G \times F$ we introduce an equivalence: $(g, f) \sim (gh^{-1}, hf)$ $f \in F$, $h \in H$. The factor space $E = G \times F/\sim$ is traditionally denoted by $G \times_H F$.

Let α denote the projection $(g, f) \mapsto g$ and $\pi_1\colon G \to E$ and $\pi_2\colon G \to G/H$ the orbit maps. We define $\Pi\colon E \to G/H$ by the commutativity of the following diagram:

$$\begin{array}{ccc} G \times F & \xrightarrow{\quad\alpha\quad} & G \\ {\scriptstyle \pi_1}\big\downarrow & & \big\downarrow{\scriptstyle \pi_2} \\ E & \xrightarrow[\quad\Pi\quad]{} & G/H. \end{array}$$

E becomes a vector bundle over G/H with a fibre F and the projection Π, the so-called *induced bundle*. The right action

$$(G \times F) \times G \to G \times F, \qquad (g, f)g_1 = ({g_1}^{-1}g, f)$$

induces a *right* G-action $(a, g_1) \mapsto ag_1\colon E \times G \to G$.

Let $\mathbf{S}$ denote the vector space of all C^∞-sections of the induced bundle. Now we are ready to define an *induced representation* $\mathbf{U}\colon \mathbf{G} \to \mathrm{Hom}(\mathbf{S}, \mathbf{S})$,

$$\mathbf{U}_g(s)(x) = \big(s(g^{-1}x)\big)g^{-1}. \tag{1}$$

The linear operator $S(g,x)$ in Theorem 4.1 of [PS-I] is then defined for each $g \in G$, $x \in X$. It is a bijection from the fibre over $g^{-1}x$ onto that over x. The initial F is identified with $\Pi^{-1}(x_0)$, $x_0 = H \in X = G/H$.

Remark 1. The notion of induced representation remains somewhat vague in [PS-I, p.99], and no reference is mentioned. It is clear from SEGAL's other publications that MACKEY's concept of induced representation is meant whose construction I have just reproduced, mainly drawing from [Ki].

Remark 2. [PS-I, pp.98–99]. If one uses the apparatus of parallelization, then it is possible to replace the space $\mathbf{S}$ by the space of all smooth functions with values in a *fixed* vector space F. The conventional treatment of physical fields over $\mathbf{M}_0$ may be regarded as based on the natural identification of *spin spaces* (i.e., spaces of *inducing representations*, see below) over different points of $\mathbf{M}_0$ that derives from the action of the vector group on $\mathbf{M}_0$. In the case of $\mathbf{M}$ it is convenient to use also the *left curved* and the *right curved* parallelizations, both derived from the representation of $\mathbf{M}$ as universal cover of $\mathbf{U}(\mathbf{2})$ w.r.t. action of $\mathbf{M}$ on itself.

The following paragraphs reproduce with minor modifications several constructions and statements from [PS-I].

The situation there is more structured since the initial homogeneous space is a Lie subgroup N of $\mathbf{G}$. We denote by ϕ the action of $\mathbf{G}$ on N which we considered earlier. It is assumed that $\phi(x)y = xy$ whenever both x and y are in N. The "parallelization map" L sends an abstract section Ψ to the vector-valued function ψ on N:

$$\psi(x) = S^{-1}(xx_0^{-1}, x)\Psi(x).$$

While $\mathbf{U}$ stood for the induced action on non-parallelized sections, the "parallelized action" U is now defined by

$$U(g) = L\mathbf{U}(g)L^{-1}.$$

It turns out that for some $g^* \in H$, using the notation $R(h)f = hf$ we have

$$U(g)(\psi)(x) = R(g^*)\psi(\phi(g^{-1})x). \tag{2}$$

Theorem 4.1. *Let R be a finite-dimensional representation of the stability subgroup $\mathbf{P}$ at the point $x_0 \in N$. Then (2) holds for the R-induced action of $\mathbf{G}$ with*

$$g^* = x_0x^{-1}g(\phi(g^{-1})x)x_0^{-1}.$$

Remark 3. There are other approaches to induced actions, see e.g. [PaSc], where the induced representation is produced on a space of vector-valued functions from the start; but that approach seems more remote from the basic ingredients of the subject.

Corollary 4.1.1. *Theorem 4.1 holds with any of the following modifications in the hypotheses:*

(i) The representation R may be projective,

(ii) The groups $\mathbf{G}$ and N may be local, and R a local representation of $\mathbf{G}_{x_0}$.

(iii) The data, i.e., $\mathbf{G}, \mathbf{N}, \mathbf{G}_{\mathbf{x_0}}$, and the action, are given purely infinitesimally.

(iv) The same as (iii) when the connected components of $\mathbf{G}, \mathbf{N}$, and R are given purely infinitesimally, but also discrete elements of $\mathbf{G}$ (forming a finite group modulo the connected component) are given that are in $\mathbf{G}_{x_0}$, and fix the unit of N. □

The notation $g \equiv g^1$ for the elements from the covering group of $\mathbf{G}_{adj}$ means that $g^1 = cg$ for some element c from the center of the covering group. In Corollary 4.1.2 of [PS-I] it is computed that, , in particular, in the notation of the Theorem,

$$g^* \equiv (\det ZW^{-1})^{1/4} \begin{pmatrix} Z^{-1}AW & -Z^{-1}B \\ -CW & D \end{pmatrix},$$

where

$$W = (A'Z + B')(C'Z + D')^{-1};$$

$$g^{-1} = \begin{pmatrix} A' & B' \\ C' & D' \end{pmatrix}.$$

In Corollary 4.1.3 of [PS-I] g^* is computed for various discrete elements involved. In the subsequent statement Corollary 4.1.4, g^* is calculated for $g \in \widetilde{\mathbf{K}}$. One of the following statements (Corollary 4.1.6) introduces the notion of the *internal* $\mathbf{Y}$ for $\mathbf{X} \in \mathcal{G}$. This notion depends on the parallelization chosen and simplifies the treatment since the infinitesimal version (i.e., the differential of the action in the representation space) dU of the induced representation takes the form

$$dU(\mathbf{X}) = -\mathbf{X} + \mathbf{r}(\mathbf{Y}),$$

where X is the vector field corresponding to $\mathbf{X}$, and r is the differential of the inducing representation R. In [ØrSe] the $r(\mathbf{Y})$ is called the *infinitesimal multiplier*. The parallelization fixed, $\mathbf{Y}$ depends on $\mathbf{X}$ and on the point Z of the underlying manifold. Table VI of [PS-I] gives the internals for $\mathbf{X}_m$ and $\mathbf{L}_{sm}$, in case of left parallelization, whence $\mathbf{Y}$ equals zero for the generators $\mathbf{Y}_m$ of the left translations on $\mathbf{U}(2)$.

4.2. Comparison of Parallelizations

It is sometimes useful to use two or more parallelizations of the given space of an induced representation. For example, the action of $\mathbf{K}$ appears simple in terms of the curved (left) parallelization; but the action of the Poincaré group is relatively complicated in this parallelization, and simple in a flat parallelization, which is valid locally but not globally on $\mathbf{M}$. The theorems of the corresponding section in [PS-I] give the connection between *the states* (as vectors in representation space) when represented in terms of two different parallelizations, global or local. The reader is referred to [PS-I] for explicit statements obtained therein (see also §5.1).

In [Sv] the "oscillator parallelization" of the induced scalar bundle ($\to$ 5.1) is considered.

5. Spaces of Scalar Representations

5.1. Scalar Representations of $SU(2,2)$

Scala representations are representations which are induced by one-dimensional representations of an isotropy subgroup. By §2.2 the isotropy subgroup of the point $p = (\pi, \mathbf{1})$ in $\widetilde{\mathbf{U}}(2)$ is isomorphic to

$$\widetilde{\mathbf{P}} \cong \mathbf{H} \rtimes (\mathbb{R}^1 \times SL(2, \mathbb{C})).$$

A one-dimensional representation of $\widetilde{\mathbf{G}}_p$ has the form

$$R_\omega(\chi((t, L), \mathbf{F})) = \exp(\omega t)$$

for a unique complex number ω, where χ is the isomorphism of $\widetilde{\mathbf{P}}$ onto $\widetilde{\mathbf{G}}_p$ introduced in §2.2.

A representation induced from R_ω is said to have *conformal weight* ω.

Remark 1. This notion plays an important part in the studies of conformal bundles. A representation R of $\widetilde{\mathbf{P}}$ is said to be of *(conformal) weight* w, w being a given complex number, in case $R(S_\lambda) = \lambda^w \mathbf{I}$ for λ real and positive, S_λ denoting the transformation $x \mapsto \lambda x$ in Minkowski space.

Theorem 5.1 [PS-I]. *In terms of the left parallelization the scalar representation of weight* ω *takes the form*

$$U(g) : \psi \mapsto \eta$$

with

$$\eta(Z) = |\det(C'Z + D')|^{-\omega} \psi(g^{-1}Z)$$

where g^{-1} *from* $\mathbf{G}$ *equals* $\begin{pmatrix} A' & B' \\ C' & D' \end{pmatrix}$. □

In the remainder of this section the scalar representation is treated in terms of the flat parallelization. Left- and flat-parallelized sections of an (abstract) section Ψ are denoted as ψ and ψ_0. I recall that $\mathbf{G}$ stands for $SU(2,2)$ and denote its left action on $\mathbf{U}(2)$ by $Z \mapsto gZ$.

Theorem 5.2 [PS-I]. *Assume that Z is obtained from $h \in \mathbf{H}$ via caley the transformation ($\to$ 2.2), and*

$$\Omega^{-1} g^{-1} \Omega = \begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix}.$$

Then the relation

$$\psi_0(Z) = p^\omega \psi(Z)$$

holds with $p = (u_{-1} + u_4)/2$. The scalar representation of weight ω acts

$$U(g)(\psi_0)(Z) = \left(\det\left(\frac{C_1 h}{2} + D_1\right)\right)^{-\omega} \psi_0(g^{-1}Z), \qquad \square$$

5.2. Covariance of Wave Operators

Theorem 5.3 [PS-I]. *In the scalar representation of weight $\omega = 1$ the equality*

$$[dU(\mathbf{L}_c + \mathbf{1}), dU(\mathbf{S})] = \mathbf{-2u_{-1}u_4dU(L_c+1)}$$

holds. If $\omega \neq 1$, the left hand side is not the product of a function with $dU_\omega(\mathbf{L}_c + \mathbf{1})$. $\square$

Corollary 5.3.1 (5.3.2 of [PS-I]).

$$dU(\mathbf{L}_f) = p^2 dU(\mathbf{L}_c + \mathbf{1}) \tag{1}$$

for $\omega = 1$. $\square$

This means, in particular, that the space $\mathbf{S}$ admits an interesting invariant subspace that may be correlated with solutions of the wave equation. On the one hand, it consists precisely of all sections annihilated by the flat wave operator $dU(\mathbf{L}_f)$. Note that this operator acts on all of $\mathbf{M}$, and not merely on the submanifold $\mathbf{M}_0$, on which it coincides with the usual D'Alembertian. On the other hand, it can be described equivalently as the space of sections annihilated by $dU(\mathbf{L}_c + \mathbf{1})$. The relation (1) is referred to in [Se87] as an example of a *bundle-invariant* property. It is also noted there that $\mathbf{L}_c + \mathbf{1}$ does not quite correspond to the usual wave operator in the Einstein universe but does give the temporal evolution that is the main purpose of the wave operator to define. This evolution is a special case of the $\widetilde{\mathbf{G}}$-transformational properties.

The situation is similar for the Dirac, Maxwell, and so-called higher spin equations, which (in their "massless" forms) correspond to irreducibly invariant, unitarisable subspaces of the section spaces of bundles induced from other representations of $\widetilde{\mathbf{P}}$ that are trivial on the translations, and are holomorphic on the homogeneous Lorentz group cover, realized as $SL(2,\mathbb{C})$, and have a uniquely determined conformal weight [Se87].

For the $w = 1$ scalar bundle the following hermitian forms are of particular importance:

$$\langle\langle a, b\rangle\rangle = \int((dU(\mathbf{L}_c) + \mathbf{1})\mathbf{a})\bar{\mathbf{b}}\,\mathbf{d_4u}, \tag{2}$$

$$\langle a, b\rangle = -i\int \dot{a}\bar{b} + i\int a\dot{\bar{b}}. \tag{3}$$

In the expression above, $\dot{a}$ stands for $X_0 a$, integration in (2) (respectively, in (3)) is over $\bar{\bar{\mathbf{M}}}$ (respectively, over $\mathbb{S}^3$); the sections a, b are supposed to be left-parallelized. In (3) each of the a, b satisfies an additional condition of annihilation by the wave curved operator.

Theorem 5.4. *$\langle\langle\cdot,\cdot\rangle\rangle$ and $\langle\cdot,\cdot\rangle$ are invariant under the representation U of $\mathbf{G}$.*$\square$

One of the next important steps is intertwining the (global) forms (2) and (3) with the usual (local) forms on Minkowski space, expressed in the flat parallelization, and determination of the curved and flat energies in terms of Cauchy data (the reader is referred to Theorem 5.6 of [PS-I]).

The proof that these hermitian forms actually become definite on distinctive positive- and negative-energy invariant subspaces ($\to$ 5.3) is not reproduced here. It is mentioned in [PS-I] that such a proof seems to require a special ($\mathbf{K}$-finite) basis or the use of Fourier transforms and certain integral special function identities.

5.3. Factors of Composition Series

Among conformal scalar bundles those of weights $w = 1$ ($\to$ 5.2) and $w = 2$ ($\to$ 6.2) are of greatest importance. The determination of factors and the order in which they occur in the corresponding composition series are important in applications ($\to$ 6.1).

It is possible to treat the situation in terms of the corresponding representations of the Lie algebra $\mathcal{G}$ on the $\mathbf{K}$-finite vectors ([PS-I, pp.135–138], where the "restricted" section space $\mathbf{E}$ of the scalar conformal bundle of weight $w = 1$ is introduced). This $\mathbf{E}$ equals the direct sum of all $\mathbf{E}_\lambda$, the evolved $\lambda \in [0, 2)$ issues from the concrete basis considerations which we skip. It is convenient to set $\mathbf{E}_{\lambda+2} = \mathbf{E}_\lambda$ for all real λ.

The factors in case $w = 1$ are the following:

Theorem 5.5. *dU is irreducible on each $\mathbf{E}_\lambda$ with $\lambda \neq 1$. $\mathbf{E}_1$ has two minimal invariant subspaces W_+ and W_-. The vector space $W_+ + W_-$ is the kernel of $dU(\mathbf{L}_c + \mathbf{1})$ (or equivalently $dU(\mathbf{L}_f)$) in $\mathbf{E}$. The chronometric Hamiltonian $H = idU(\mathbf{X}_0)$ is positive (negative) on W_+ (respectively, W_-). The restriction of U to (the closures of) W_+ and W_- are unitary, with the unitary structures $\langle\cdot,\cdot\rangle$ and $-\langle\cdot,\cdot\rangle$, respectively, (defined in §5.2). $\mathbf{E}_1$ is the sum (not direct) of the invariant* *subspaces* V_+, V, V_-. *Moreover,* $V \cap V_+ = W_+$, $V \cap V_- = W_-$, *and* $V_+ \cap V_- = 0$. *The factor spaces* V_+/W_+, V_-/W_-, *and* $V/(W_+ + W_-)$ *are irreducible and unitarizable, with the unitary structures* $-\langle\langle\cdot,\cdot\rangle\rangle$, $-\langle\langle\cdot,\cdot\rangle\rangle$, *and* $\langle\langle\cdot,\cdot\rangle\rangle$, *respectively. The spaces* V_+, V_-, *and* V *have no* $\mathcal{G}$*-invariant complements for* W_+ W_-, *and* $W_+ + W_-$, *respectively.* □

For $w = 2$ the space $\mathbf{E}_2$ is the only subspace of $\mathbf{E}$ which is not irreducible. $\mathbf{E}_2$ equals the direct sum of the spaces V_+, V, and V_- which are irreducible and invariant. The chronometric Hamiltonian H is positive (respectively: negative) on V_+ (respectively: on V_-). For the proofs see [PS-I], [Mo90], and references therein. There the subspaces involved are determined explicetely in terms of the $\mathbf{K}$-finite basis. The latter is labelled by quantum numbers ($\rightarrow$ 6.4) associated with a system of subgroups $O(2) \subset O(3) \subset O(4)$ of $\mathbf{K}$, together with the $O(2)$ subgroup generated by $\mathbf{X}_0$. This basis is used also in the treatment of higher spin representations, and is important in physical applications. The determination of factors of scalar bundles can be used later for representations of spannor ($\rightarrow$ 6.2) and plyor ($\rightarrow$ 6.3) bundles as has been already done in case of two-dimensional chronometry [ØrSe].

6. Elementary Particles Associations

6.1. The General Viewpoint

I.E.Segal introduces in [Se91] the notion of a *clan* (consisting of all fields on $\mathbf{M}$ having designated transformation properties under $\widetilde{\mathbf{G}}$) and considers the *fermionic* ($\rightarrow$ 6.2) and *bosonic* ($\rightarrow$ 6.3) ones. He describes the part of his program as an extension of Wigner's classical formulation of relativistic particles as irreducible unitary positive-energy representations of the Poincare group $\widetilde{\mathbf{P}}_0$ to one in which the causal group $\widetilde{\mathbf{G}}$ of $\mathbf{M}$ is substituted for the $\widetilde{\mathbf{P}}_0$. I.E.Segal emphasizes the fact that the bundle or, equivalently, the transformation group aspect, is no less essential than the pure group representation aspect, and physically more fundamental. From hereon the use of some terminology from elementary particle physics seems inevitable: see my Remark 3 in §6.4. The spatiotemporal labelling of vectors in the induced representation spaces is necessary for the concept of *local interaction*

to be meaningful, and effectively necessary for the treatment of the closely related issue of causality. An example is the difference between the natural models for the electron ν_e and muon ν_μ neutrinos that emerge [P-IV, PS87, Se91].

We discuss these notions more explicitly. The starting point is an induced representation $\mathbf{U}$ of $\widetilde{\mathbf{G}}$ ($\rightarrow$ 4.1) with $\mathbf{V}$ as the representation space. Only representations with a *composition series* are considered [Se91], i.e., with a maximal chain of invariant subspaces

$$\mathbf{0} \subset \mathbf{S_0} \subset \mathbf{S_1} \subset ... \subset \mathbf{S_n} = \mathbf{V}. \tag{1}$$

A *subquotient* of $\mathbf{U}$ is defined as the corresponding representation on the quotient space $\mathbf{S}/\mathbf{T}$ between invariant subspaces $\mathbf{T} \subset \mathbf{S} \subset \mathbf{V}$. The *factors* are those subquotients that are irreducible, i.e., for which $\mathbf{T}$ is a maximal invariant subspace of $\mathbf{S}$. I.E.Segal distinguishes between an *exact particle*, which is represented by a vector in the clan and corresponds to a free physical state and a *reduced particle*, a theoretical entity which is extracted from a clan by formation of subquotients. The factors define the *elementary particle spectrum;* the *stable spectrum* consists of those factors that are unitary and have a one-sided *frequency spectrum* (i.e., the one-sided spectrum of the chronometric Hamiltonian H ($\rightarrow$ 3.8)). The *(chronometric) energy* of a particle in the (normed) state f is defined as $\langle Hf, f\rangle$ where $\langle\cdot,\cdot\rangle$ is the positive-definite Hermitian form in the corresponding factor. Although there will in general be many inequivalent (non-conjugate) chains (1), the factors are unique as group representations. Notwithstanding the lack of uniqueness for the maximal chain, there are nontrivial constraints on the order in which the factors occur, corresponding to the order of inclusion of the corresponding invariant subspaces. I.E.Segal remarks [Se91] that this contrasts greatly with the entirely arbitrary order in which the factors occur in the case of a fully decomposable representation, as in conventional theory. Thus, in the chronometric fermion clan ($\rightarrow$ 6.2), the exon x appears as a bottom invariant subspace, or factor, and the electron e as a top factor; in the middle are the muon and the electron neutrino factors, in that order. In the boson clan ($\rightarrow$ 6.3), the photon appears as a bottom factor, above which are the bare versions of W and Z.

Corresponding to any given chronometric clan is a relativistic free particle family consisting of the direct sum of the stable factors, restricted to the conventional Poincare group $\mathbf{P}_0$ and fixed in mass ($\rightarrow$ 6.2). Because of indecomposability, the action of $\mathbf{P}_0$ on the clan mixes up the factors, and so is quite different from the relativistic action of $\mathbf{P}_0$ on the direct sum of the stable factors. In other words, the chronometric free temporal evolution gives rise to apparent particle production within the frame of the relativistic limit. It is called [Se91] *indecomposable* production, to distinguish it from *Lagrangian* production of the conventional type; both are causal and covariant. Since indecomposable production is absent in the relativistic limit of the chronometric theory, it appears as a weak interaction in conventional terms. But there are also Lagrangian interactions between neutrinos and other particles ($\rightarrow$ 6.4), which would be classified as weak in the relativistic theory.

6.2. The Fermion Clan

The corresponding representation is *induced* ($\rightarrow$ 4.1) to $\widetilde{\mathbf{G}}$ ($\rightarrow$ 2.2) from the *spannor* representation Σ of $\widetilde{\mathbf{P}}$ ($\rightarrow$ 2.1). The representation Σ *is defined* as the direct sum of Σ^+ and of Σ^-, where

$$\Sigma^+(g) = (\det T)^2 \begin{pmatrix} T & (i/2)FT^{*-1} \\ \mathbf{0} & T^{*-1} \end{pmatrix},$$

and Σ^- may then be defined, within equivalence, as either the parity transform P ([P-IV]) of Σ^+ or the complex conjugate representation. The matrix T above stands for $e^{t/2}L$ where $(t, L, F) \in \widetilde{\mathbf{P}}$, see §2.1.

Remark 1. I drop the index $d = 2$ from the appropriate notation of [P-IV], d being the *degree* of the spannor.

Each of the two inducing representations is defined in $\mathbb{C}^4$. In $\mathbb{C}^8 = \mathbb{C}^4 \bigoplus \mathbb{C}^4$ the discrete symmetries C, P, T act as well [P-IV, Theorem 16.3.1]. The eight-dimensional *spin representation* Ξ of $\widetilde{\mathbf{G}}$ is introduced in [P-IV]. It figures in the following useful statement (where the *spannor bundle* stands for the corresponding *induced bundle* ($\rightarrow$ 4.1)).

Theorem 6.1 (Corollary 16.4.4 of [P-IV]). *The spannor bundle is (bundle-wise) the tensor product of the scalar bundle* ($\rightarrow$ 5.1) *of weight* $w = 2$ *with the spin representation of* $\widetilde{\mathbf{G}}$. □

The rigorous mathematical derivation of the corresponding elementary particles seems to be still absent in the literature. A fundumental attempt has been made in [P-IV] but later it was noted [Se91] that "the composition series shown in [P-IV] is inexact and should be replaced by that indicated in [Se91]". The latter presents the fermion clan as the direct sum of a stable subspace and a *tachionic* subspace (i.e., one with the both-sided unbounded frequency spectrum). The stable subspace is the direct sum of a positive-frequency ($\rightarrow$ 6.1.) representation F^+ (which is "indecomposably built" into the representation induced from Σ^+) with its complex conjugate F^-. Physical assignments for the reduced ($\rightarrow$ 6.1) particles are given in Table 1 of [Se91]; those for *antiparticles* are obtained by interchanging the left and right *spins* [PS-II].

There are exactly four factors (hence, the fermionic clan includes four particles and four antiparticles), they correspond to the *exon* x, *muon* ν_μ and *electron* ν_e *neutrinos*, and *electron* e [SeØr]. This physical particle assignment is depicted in Column I of the mentioned table. The assignment is determined by the *massive/massless* character of the particle and the vanishing/nonvanishing of its interaction with the photon ($\rightarrow$ 6.4). The massive (respectively: massless) means in the context that the *Gelfand-Kirillov dimension* equals four (respectively: three), see Column VII of the table. The second column is the *bare* (*chronometric, intrinsic*) *mass* of the particle expressed in chronometric units. It is defined as the minimum of the chronometric energy ($\rightarrow$ 6.1) in the corresponding factor and equals $3/2$ for the neutrinos, $5/2$ for exon and electron; this is the contents of Column II. It is remarked in [Se91] that the minimum of the Minkowski energy vanishes and that this bare mass is far below the level of physical observability, since the proton ($\rightarrow$ 6.4) *relativistic mass* $m_p \approx 10^{40}$. This latter notion is introduced as follows. The starting point is an exact particle represented by an eigenstate of the chronometric Hamiltonian H ($\rightarrow$ 3.8). Let M^2 denote the usual relativistic mass operator $T_0^2 - T_1^2 - T_2^2 - T_3^2$. This operator and H act on the representation space; then the state in question has to be an approximate eigenstate for

$$e^{-itH}M^2e^{itH}$$

of the same narrow-width eigenvalue, for a nongenerically long time interval. Thus in particular, $[M^2, H]$ should have expectation value 0. The states with such constraints appear likely to exist and their relativistic masses be computable from them. One of the next *quantum numbers* is the *height* of the particle which seems to correspond to the order of inclusion of the corresponding invariant subspace. The four just mentioned particles have heights from 1 to 4, in the same order.

Remark 2. The spannor section space of the two-dimensional chronometry has been treated in [ØrSe]. It has much in common with the physical four-dimensional situation. The investigation of the composition series exploits effectively the analog of the above stated Theorem 6.1.

6.3. The Boson Clan

It is induced ($\to$ 4.1) from the particular 15-dimensional indecomposable representation (see [P-IV]) of the Poincare group $\widetilde{\mathbf{P}}$. The corresponding *field* (i.e., section of the induced bundle or vector in the space of induced representation) is referred to as *plyor* (see [P-IV] and several later publications).

Let me reproduce the corresponding information from [P-IV]. To characterise the plyor representation infinitesimally (see Lemma 17.1.1. of [P-IV]) it is convenient to use 8×8 matrices $w_m, m = -1, 0, \ldots, 4$, therein introduced. The two subspaces P_+, P_- of *conformal weight* 1 ($\to$ 5.1) include photons as reduced *vector* particles. P_+ is defined by the basis $(w_{-1} - w_4)w_s$, and P_- is similarly defined by $(w_{-1} - w_4)w_s w_j w_k$; all indices have values from 0 to 3. The bases are chosen in the *spin space* ($\to$ 4.1).

The corresponding subspaces of the conformal weights 0 and -1 are similarly defined by their bases in the spin space [P-IV].

In the statement below π stands for the infinitesimal plyor representation and the generators $\mathbf{T}_m$ (of time and space translations in the Minkowski world $\mathbf{M}_0$) have been distinguished earlier ($\to$ 3.8).

Theorem 6.2. *The total space of plyors is indecomposable under the action of* $\widetilde{\mathbf{P}}$. *On restriction to the scale-extended Lorentz group* ($\to$ 2.1), *it decomposes as direct sum (of* $w = 1, 0, -1$ *subspaces) shown in Table 17.1.1 of [P-IV]. Its subspace of weight* -1 *leaks nontrivially into that of weight* 0, *and that of weight* 0 *into that of weight* 1, *while the latter subspace is* $\widetilde{\mathbf{P}}$*-invariant.* (*The exact meaning of leaking is that, f.e., the* $w = 0$ *subspace is taken by the operators* $\pi(\mathbf{T}_m)$ *into the* $w = 1$ *one; etc.*) □

6.4. The Chronometric Fermion-Boson Interaction

The exposition of this section is mainly extracted from [Se91]. I add references to make the reading easier.

Remark 3. There are several standard notations and terminology from elementary particle physics somewhere above and in the remainder of this section. The reader is to consult one of the numerous books on the subject (see, f.e., [Derd] and references therein).

The *interaction Lagrangian* L_I is the essentially unique $\widetilde{\mathbf{G}}$-invariant *coupling* of the boson clan with the local bilinear fermion clan *current* (bilinear current here stands for the section of the tensor product of the bundle with its dual [PS-II]).

Charged particles are either electrons ($\rightarrow$ 6.2) or composites with electrons. (*A proton* p, f.e., is chronometrically modelled as $p = x + e^{+} + \nu_e$).

If f denotes the fermion and A, the boson state, where A is represented canonically by a matrix on the fermion spin space [P-IV], then

$$L_I(f, A) = \int \langle\langle Af, f\rangle\rangle \, d_4x$$

where the inner product is the *invariant* [P-IV] one in the fermion spin space at x, and for the measure d_4x see §3.7. The bosons have weights dual to those of the fermion currents: since these are of weights $3/2+3/2, 3/2+5/2, 5/2+5/2$ (Table 1 of [Se91]), the boson weights are $1, 0$, and -1 (the sum of the three weights must equal 4, [PS-II]). The weigts 0 and -1 are well defined only in the the relativistic limit; $w = -1$ states leak ($\rightarrow$ 6.3) under the action of $\widetilde{\mathbf{G}}$ into $w = 0$ states, and $w = 0$ states leak similarly into the $w = 1$ subspace, which is $\widetilde{\mathbf{G}}$-invariant. Corresponding to the three different types of currents just indicated, there are *three different types of interactions*, in terms of relativistic limit.

(i) Two $w = 3/2$ fermions and a $w = 1$ boson:

The two $w = 3/2$ fermions are electrons and neutrinos. The $w = 1$ bosons include the photon, at the bottom of the subspace, and distinct candidates for the bare $W = W_0$ and the Z, the former in a neutral form (the physical W^{+}, W^{-} being composites of W_0 with electrons and other particles). All three reduced particles have distinct quantum numbers that play a role comparable to the gauge degrees of freedom in the standard model. Charges of the $w = 3/2$ particles are included automatically in the form of the Lagrangian; e.g., the neutrino-photon integrated interaction vanishes, as a consequence of the transformation laws (or equivalently, the Dirac and Maxwell equations). The neutrino interactions with the W_0 and the Z are nonvanishing and parallel those of e with the latter, providing a form of weak isospin.

(ii) A $w = 3/2$ fermion, a $w = 5/2$ fermion, and $w = 0$ boson:

This is not readily characterized in relativistic terms but seems to underlie low-energy-electron and top-neutrino ("top" - in terms of the chain (1), see §6.2) interactions with baryons and light mesons. The large nucleon to physical electron mass ratio appears to give this interaction a *strong* appearance in relativistic terms, although in bare chronometric terms it appears formally as approximately symmetric between the e and the x. The $w = 0$ sector includes a natural candidate for the neutral pion, whose decay into two photons may derive primarily from the leaking of the $w = 0$ bosons into the $w = 1$ subspace. The decay into neutral pions of the K^0 may be of similar character.

(iii) Two $w = 5/2$ fermions and a $w = -1$ boson:

This interaction appears as purely strong in relativistic terms. The stable reduced elementary boson in this sector shows mixing of two relativistically invariant components and would be expected to leak into $w = 0$ bosons, among other possible decays. This suggests identification with the K^0, but the mixing shown by the B^0 and the D^0, together with their decay products, suggests they may be higher forms of the K^0 via the above proposed mechanism. The top positions of the e and the K^0 in their respective clans should facilitate this mechanism. The conformal weight sum constraint suppresses decay of the K^0 into $\mu^+\mu^-$ but allows $K^0 \to x\bar{x}$.

There are several other claims in [Se91] as regards chronometric description of elementary particles characteristics and interactions. I.E.Segal argues, in particular, that all relativistically "internal" symmetries may originate in the interplay between the quantum numbers associated with the maximal subgroups $\mathbf{K}$ and $\mathbf{P}$ of $\widetilde{\mathbf{G}}$ in the chronometric clans and thus be of an ultimately geometrical character.

7. Chronometry and Extragalactic Astronomy

It is outlined in [Se 91] (and is seen in the examples of several previous sections) that the chronometric treatment of microscopic phenomena (elementary particles, interactions, quantization) is not superradical in comparison with the standard model. The situation in extragalactic astronomy is quite different since in its main predictions it contrasts greatly with the now-a-days most accepted Friedman-Lemaitre cosmology.

I.E.Segal collaborates in the subject with J.F.Nicoll. During the past decades more than twenty papers appeared in astronomical and physics journals in which the predictions of the chronometric theory and systematic astronomical observation were compared in detail [Se91, SeNi, and references therein].

One particular difference betweeen the two cosmological theories is the chronometric redshift-distance relation

$$z = \tan^2(r/2), \tag{1}$$

where r is the distance in radians on the sphere $\mathbb{S}^3$ that represents space ($\to \mathbf{1}$).

The equation (1) has been obtained in [Se76] at the classical quantum mechanical level under particular assumptions on the photon wave function. In [SZ] this law is rederived on a mathematically more rigorous basis for a photon of localized spatial support. The unitary representation of the conformal group on the Hilbert space of normalizable photon wave functions is applied, in the Schrödinger and

Heisenberg representations. The analysis shows also the existence of photon states of cosmic spatial support that are not redshifted at all, as time evolves.

Recall the crucial idea in deriving the law (1): According to chronometry, the "true" Hamiltonian is the operator H ($\to$ 3.8) corresponding to the advance of chronometric time t ($\to$ 2.2), while direct laboratory observations of the energy yield only the scale-covariant component H_0 ($\to$ 3.8) of H. This component does not commute with H and so is not conserved; after an elapsed chronometric time s, it is represented, in the Heisenberg picture, by the operator

$$H_0(s) = e^{-isH} H_0 e^{isH}.$$

The redshift z is defined so that $1+z$ is the factor by which the special relativistic energy is reduced in the state in question. Chronometry explains, intrinsically, why the redshift is "red", though it may appear to lack the intuitive simplicity of a Doppler explanation for the redshift, which has become so familiar as to appear almost axiomatic to some astrophysicists.

The distance r in (1) is not an observable quantity, but the purely geometric relations between apparent luminosity and distance (as well as other observed quantities, such as angular diameter) permit the distance to be eliminated and purely observable relations derived. These relations are then tested on the large samples of galaxies, quasars, and radio sources. Remarkably good agreement between predictions and observation is found. Chronometry leads to a much better fit with observation than do Friedman-Lemaitre models with their two free parameters q_0 and Λ. In addition, a number of anomalies within the Friedman-model of cosmology are simply eliminated: the apparent superrelativistic lateral velocities of a number of sources, and extraordinary luminosity and apparent evolution of quasars. The cosmic background radiation is not necessarily indicating the "big bang" but is predicted as the temporally homogeneous equilibrium photon gas established by the diffusion and scattering of electromagnetic radiation around the physical space $\mathbb{S}^3$ in accordance with energy conservation.

In [Se91] it is also argued that the mechanism of indecomposable production ($\to 6.1, 6.3$)

$$\nu_e \to \nu_e + \nu_\mu + \tilde{\nu}_\mu$$

may contribute to the solar neutrino deficiency caused by the attrition of the number of ν_e particles in flight due to the conversion into ν_μ pairs that are unable to revert to ν_e. On the other hand, the inverse process can proceed on a comparable scale only by Lagrangian rather than indecomposable production. As regards gravitation, chronometry says that there is no special force of gravity as such: it is simply the totality of the scale-contravariant [Se86], or super-relativistic, components of the energies associated with forces that also act microscopically ($\to$ 6.4).

It is stated in [Se91] (see references therein) that cosmic ray observations have been indicative of a neutral extremely long-lived hadron-like particle coming from Cygnus X-3 and from Hercules X-1. The chronometric exon x ($\to$ 6.2) is a theoretical counterpart for these particles. Its relativistic mass ($\to$ 6.2) varies, in principle. If it is of the order of the neutron mass, confusion between x and n could be a factor in the many revisions in the estimated neutron lifetime in recent decades and an anomaly in neutron scattering [Sla] but observations on Hercules X-1 suggest that it may be light enough to be confused with a neutrino, if produced in high-energy collisions. Further cosmic ray observations are needed, but conclusive identification of the cygnet with the exon will depend on an observation of the latter in accelerator experiments. It should be possible to produce it in energetic electron-nucleon or nucleon-nucleon collisions.

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