"Russian Troika" as the New Spatio-Temporal Paradigm.

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Abstract

It is proved that the (local) causal structure of the (flat) Minkowski space-time M_0 can be (metrically) defined by each of the three (curved) worlds M, F, L (and there are no other options to represent M_0 in such a way). M, F, L are the respective Lie groups supplied by a suitable bi-invariant Lorentzian metric: $m = u(2), f = u(1,1), l = osc$ as Lie algebras.

The three worlds might be viewed as (the most symmetric) general relativistic space-times. They are supposed to substitute M_0 (like a haudred years ago the Newtonian world had to give up its leading role when M_0 emerged). Now, there are three Hamiltonians to drive evolution of a physical system (a "Russian Troika").

The findings seem to set up quite a new prospective to develop the "Particles and their Interactions" theory.

Key words and phrases: symmetric spaces, relativity, Segal's chronometric theory, quantum-mechanical hidden variables.

Main Part

Let M_0 stand for the Minkowski space-time, M is for the unitary group $U(2)$.

The image $c(M_0)$ of the Caley map c ([Se76], [Le95]) is an open dense subset of M. The family $\{C_y\}$ is in M_0 but it also determines a bi-invariant cone field on M; here $C_y = y + C$, C is the light cone in Minkowski space-time.

Future sets are canonically defined in a universal cover $\tilde{M}$. Recall the (fractional-linear) G-action on M :

$$g(z) = (Az + B)(Cz + D)^{-1}$$

here an element g (with 2 by 2 blocks A, B, C, D) is from $G = SU(2, 2)$. This action lifts canonically to a $\tilde{G}$ -action on $\tilde{M}$ (which preserves the causal structure). Proofs of all the above statements can be found in [Se76, PaSe82a].

Theorem 1 ([Ad76], [Se76]). *If a bijection f of $\tilde{M}$ preserves the causal structure, then f is defined by some $\tilde{g}$ from $\tilde{G}$.*

It is known ([PaSe82a, PaSe82b]) that to model particles on $\tilde{M}$, one can start with a (compact!) world M (in view of "automatic periodicity" results).

The universal cover $\tilde{P}$ covers the (scale-extended) Poincare group P twice.

Theorem 2 ([PaSe82a]). *A stability subgroup (of an event x from $\tilde{M}$) is isomorphic to $\tilde{P}$. There is a commutative diagram (an intertwining relation between the $\tilde{P}$ action in M and $\tilde{P}$ action in M_0, the latter chosen in the form of $u(2)$).*

The following is a well-known result.

Theorem 3. *A metric on a Lie group N is bi-invariant if and only if the (respective) form in the Lie algebra n is invariant.*

Remark 1. An invariant non-degenerate form in a simple Lie algebra has to be proportional to the Cartan-Killing form.

Theorem 4 ([GuLe84]). *In dimension four, there are exactly three non-commutative Lie algebras admitting non-degenerate Lorentzian invariant form: $m = u(2), f = u(1, 1), l = osc$.*

Remark 2. The first two cases are well-known (there are no other non-Abelian four-dimensional semi-simple Lie algebras). It was a certain surprise to find a solvable algebra in that list (it can formally be defined by the following commutation table: $[l_2, l_3] = l_1, [l_2, l_4] = l_3, [l_4, l_3] = l_2$).

Quantum-mechanical wave functions are sections of (certain) vector bundles over $\tilde{M}$; "induced bundles" since they are determined by $\tilde{G}$ representations, induced from finite-dimensional representations of $\tilde{P}$. For a scalar particle, the fiber is complex one-dimensional, etc.

In Segal's chronometry, the entire list of known particles is derived mathematically. One chronometric particle (the "exon") has not yet been experimentally identified (see [Se91] or a survey [Le95] for the details on the above).

The "architecture" of the scalar bundle is determined by a certain (conformally covariant) second order differential operator ("curved wave operator", compare to the "flat wave operator", a standard one), see [PaSe82a] and [Le04] for more details.

The scalar bundle (together with known finite-dimensional $\tilde{P}$ representations) determines higher spin bundles (see [PaSeVo87, Se98, SeVo98]).

For an explicit description of a curved wave operator, one other well-known result (adjusted to the current case) is instrumental.

Theorem 5 (see [Or81]). *In a four-dimensional conformally flat pseudo-Riemannian space of constant scalar curvature R if T is the Laplace-Beltrami operator, then $T + R/6$ is conformally covariant.*

To explicitely present the three conformally covariant wave operators, recall from [PaSe82a] the following basis $\mathbf{L_{ij}}$ (with $\mathbf{L_{ij}} = -\mathbf{L_{ji}}$) in $su(2,2)$:

$$[\mathbf{L_{im}}, \mathbf{L_{mk}}] = -e_m \mathbf{L_{ik}},$$

where the 6-tuple $(e_{-1}, e_0, e_1, e_2, e_3, e_4)$ equals $(1, 1, -1, -1, -1, -1)$.

The $G-$action results in vector fields L_{ij} (non-boldface) on M with the opposite (compared to the just stated) right sides in the commutation table. $X_0 = L_{-10}, X_1 = L_{14} - L_{23}, X_2 = L_{24} - L_{31}, X_3 = L_{34} - L_{12}$ form a left-invariant orthonormal frame on $M = U(2)$. In $M-$case, the scalar curvature is 6 (curvature computations are real quick in all cases treated here, since the metric is bi-invariant, see [Le85]). Now, the conformally covariant wave operator ("originating from the M-viewpoint") is

$$X_0{}^2 - X_1{}^2 - X_2{}^2 - X_3{}^2 + 1,$$

as given in [PaSe82a].

Globally, the world $\tilde{M}$ is R^1 times S^3, where S^3 is represented by the group $SU(2)$.

The world F is the universal cover of $U(1,1)$, it is R^4, topologically. Its relatively compact form, a four-dimensional orbit in $U(2)$, is defined by four orthonormal vector fields H_0, H_1, H_2, H_3 on $U(2)$, where $H_0 = L_{-10} - L_{12}, H_1 = -L_{-12} + L_{01}, H_2 = L_{02} - L_{-11}, H_3 = L_{34}$. It is an easy exercise to show that the four fields form an $u(1,1)-$subalgebra of $su(2,2)$. The scalar curvature is now negative two (a routine calculation),

$$(H_0)^2 - (H_1)^2 - (H_2)^2 - (H_3)^2 - 1/3$$

is one more conformally covariant wave operator.

The third world, L is R^4, topologically. Again, its relatively compact form, a four-dimensional orbit in $U(2)$, is defined by four vector fields l_1, l_2, l_3, l_4, where

$$l_1 = L_{-10} + L_{04} + L_{-11} + L_{14},\ \ l_2 = (1/2)(L_{-12} + L_{24} + 2L_{03} + 2L_{31}),$$

$l_3 = (1/2)(L_{-13}+L_{34}+2L_{02}+2L_{12}),\ l_4 = (1/8)(-5L_{-10}-3L_{-11}+3L_{04}+5L_{14}+4L_{23})$.

One can verify the (above stated) commutation table for the oscillator Lie algebra. The expression for the metric is the one shown below for the wave operator.

The scalar curvature is now zero (as shown in [Le86] where this world has been studied separately; all together the three worlds have been discussed in [Le85]).

The respective conformally covariant wave operator is

$$2l_1l_4 - (l_2)^2 - (l_3)^2.$$

Using Table I from [SeJa81], one proves that the three worlds share the same light cone at, say, the neutral element of $U(2)$ (which means at the origin of the Minkowski world). Since the action remains the same (the linear-fractional one), the three worlds have to share the same cone field globally.

Remark 3. All three Hamiltonians (a "Russian Troika") drive the (quantum-mechanical) evolution of a particle. Currently, the L- and F-aspects are yet ignored by the "standard" science (that is why "hidden variables" are mentioned in key words and phrases).

The present findings have been first made public at the special meeting of the "Newton-Einstein-Segal" Seminar (Boston University, August 30, 2003, it was 5 years since I. Segal has passed away). Later, the talk has been given at the Third International Conference "East - West on Neva Banks" (held October 9-11, 2003, at the Saint Petersburg University, Russia).

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References

[Ad76] Aleksandrov, A.D. Vestnik Leningrad. Univ. (1976), no.19 (Ser. Mat. Mech. Astr. vyp. 4), 5-28; English transl. in Vestnik Leningrad Univ. Math. 9(1981).

[GuLe84] Guts, A.K. and Levichev, A.V.; Dokl.Akad.Nauk SSSR 277(1984), 253-257; English transl. in Soviet Math. Dokl. 30(1984).

[Le85] Levichev, A.V. In: Group Theoretical Methods in Physics. Proceedings of the III International Seminar, Yurmala, May 1985, vol.1,Nauka, Moscow 1986, 145-150.

[Le86] Levichev, A.V. Siberian Math. J. 27(1986), 237-245.

[Le95] Levichev, A.V. In: Semigroups in Algebra, Geometry and Analysis,

Eds.: Hofmann/Lawson/Vinberg; Walter de Gruyter & Co., Berlin- New York (1995),77-103 (on display,http://math.bu.edu/people/levit)

[Le04] Levichev, A. V. More about the three symmetric general relativistic space-times. Sobolev Institute of Mathematics (Siberian Division of the RAS), preprint(submitted), Novosibirsk, 2004.

[Or81] Orsted, B. J. Funct. Anal. 44(1981), 1-23.

[PaSe82a] Paneitz, S.M. and Segal, I.E. J. Funct. Anal. 47(1982), 78-142.

[PaSe82b] Paneitz, S.M. and Segal, I.E. J. Funct. Anal. 49(1982), 335-414.

[PaSeVo87] Paneitz, S.M.; Segal, I.E.; Vogan, D.A., Jr. J. Funct. Anal. 75(1987), 1-57

[Se76] Segal, I. E. Mathematical Cosmology and Extragalactic Astronomy. N.-Y.: Academic, 1976.

[SeJa81] Segal, I.E.; Jakobsen, H.P.; Orsted, B.; Paneitz, S.M.; Speh, B. Proc. Nat. Acad. Sci. USA 78(1981), 5261-5265.

[Se91] Segal, I.E. Proc. Natl. Acad. Sci. USA 88(1991), 994-998.

[Se98] Segal, I.E. J. Funct. Anal. 154(1998), 542-558.

[SeVo98] Segal, I.E.; Vogan, D.A., Jr.; Zhou, Z. J. Funct. Anal. 156(1998), 252-262.