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Geroch decomposition implications for quantum gravity.

Research Article

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Abstract: Geroch decomposition for Schwarzschild case shows, that gravity may be explained as spacelike manifold minimally coupled to a scalar field. I show, that in this approach spacetime curvature may be modeled as local acceleration of photons according to regular Rindler transformation. Above brings important implications for quantum gravity theories.

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1. Introduction

They are known scalar theories of gravity on flat spacetime [1] explaining gravity as interaction between bodies.

I show, that Geroch decomposition for spherically symmetric case, opens way to model curved spacetime as effect of local acceleration of light. In this perspective gravity is described by omnipresent scalar field (equal to gravitational time dilation factor) acting on flat spacetime. To confirm it, I show in section 3.3, that there exist a way to consider local light acceleration with Rindler's transformation. In result we derive null geodesics in regular Schwarzschild metric for stationary Killing observer reference frame.

In second part of the article I also propose explanation why we observe so strong relationship between light behavior and spacetime curvature. I also conclude, that one might now consider Kerr metric with imaginary angular momentum to model elementary particles.

All may bring important implications for quantum gravity theories and opens new areas for research and generalizations.

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2. Introducing Killing vector fields

As usual, Einstein summation convention applies. Commas denote partial derivatives:

$$\varphi_{,\mu} = \frac{\partial \varphi}{\partial x^{\mu}} \tag{1}$$

Semicolons denote covariant derivative:

$$\nabla_{\mu} X^{\alpha} = X^{\alpha}_{;\mu} = X^{\alpha}_{,\mu} + \Gamma^{\alpha}_{\mu\nu} X^{\nu} \tag{2}$$

where Γ 's are the connection coefficients.The geodesic equation states that the covariant derivative of the particle four-velocity

$$u^{\mu} = \frac{dx^{\mu}}{d\tau} \tag{3}$$

along itself vanishes:

$$0 = \nabla_{\mu} u^{\alpha} = u^{\mu} u^{\alpha}_{;\mu} = \frac{dx^{\mu}}{d\tau} \left[\left(\frac{dx^{\alpha}}{d\tau} \right)_{,\mu} + \Gamma^{\alpha}_{\mu\nu} \frac{dx^{\nu}}{d\tau} \right] \tag{4}$$

$$0 = \frac{dx^{\alpha}}{d\tau^2} + \Gamma^{\alpha}_{\mu\nu} \frac{dx^{\mu}}{d\tau} \frac{dx^{\nu}}{d\tau} \tag{5}$$

Here τ is affine parameter, which is proper time for timelike geodesics. On a spacetime, a Killing vector field [2] generates an isometry of spacetime. Generally, this requires solving the equation

$$\xi_{\mu;\nu} + \xi_{\nu;\mu} = 0 \tag{6}$$

but in the case where we have a coordinate chart in which the metric coefficients are independent of a coordinate, then the vector field of that coordinate is automatically a Killing field.

For a geodesic, it defines a constant of motion, since

$$\nabla_{\mu} \left(u^{\mu} \cdot \xi \right) = u^{\nu} \left(u^{\mu} \xi_{\mu} \right)_{;\nu} = 0 \tag{7}$$

$$u^{\nu} u^{\mu}_{;\nu} \xi_{\mu} + u^{\nu} u^{\mu} \xi_{\mu;\nu} = 0 \tag{8}$$

the first term being zero because of the geodesic equation and the second term because of anti-symmetry of $\xi_{\mu;\nu}$

For the Schwarzschild spacetime

$$ds^2 = -\left(1 - \frac{r_s}{r}\right) dt^2 + \frac{dr^2}{\left(1 - \frac{r_s}{r}\right)} + r^2 \left(d\theta^2 + \sin^2\theta \, d\varphi^2\right) \tag{9}$$

we see that the metric coefficients are independent of t and φ , so that ∂_t and ∂_φ are Killing fields.

In Schwarzschild coordinates, a particle worldline has the four-velocity

$$u^{\mu} = \left[\frac{dt}{d\tau}; \frac{dr}{d\tau}; \frac{d\theta}{d\tau}; \frac{d\varphi}{d\tau}\right] \tag{10}$$

so for

$$\xi^{\mu} = [1; 0; 0; 0] \tag{11}$$

we have the dot product

$$-e_0 = g_{\mu\nu}\xi^{\mu}u^{\nu} = -\left(1 - \frac{r_s}{r}\right)\frac{dt}{d\tau} \tag{12}$$

and similarly for

$$\xi^{\mu} = [0; 0; 0; 1] \tag{13}$$

we have

$$l = r^2\left(\sin^2\theta\right)\frac{d\varphi}{d\tau} \tag{14}$$

The physical meaning is that e_0 and l are specific (per-mass) energy and azimuthal angular momentum of the particle at infinity, respectively. We could actually derive them in a variety of ways, including from the geodesic equation, which for the t-coordinate says:

$$\dot{t} = \frac{dt}{d\tau} \tag{15}$$

$$\frac{d^2t}{d\tau^2} + \frac{r_s}{r^2}\cdot\frac{1}{\left(1 - \frac{r_s}{r}\right)}\frac{dt}{d\tau}\frac{dr}{d\tau} = 0 \tag{16}$$

thus

$$\frac{d\dot{t}}{\dot{t}} = -\frac{d\left(1 - \frac{r_s}{r}\right)}{\left(1 - \frac{r_s}{r}\right)} \tag{17}$$

which straightforwardly integrable, the specific energy e coming from the constant of integration. Note that this is relativistic energy, so for free-fall from rest at infinity or an orbit as exactly escape velocity has $e_0 = 1$ Reading them off from the metric, whenever we can, is certainly easier.

3. Gravity with scalar field

3.1. Geroch decomposition

Let us apply Geroch decomposition [3] for spherically symmetric case and discuss its implications.

If we have an asymptotically flat spacetime with a timelike Killing vector field μ with norm-squared

$$\lambda = -\xi^{\mu}\xi_{\mu} \tag{18}$$

and twist

$$\omega_{\mu} = \epsilon_{\mu\nu\rho\delta}\xi^{\nu}\nabla^{\rho}\xi^{\delta} \tag{19}$$

using the tensor

$$\gamma_{\mu\nu} = \lambda g_{\mu\nu} + \xi_\mu \xi_\nu \tag{20}$$

the spacetime metric takes the form

$$ds^2 = -\lambda \left(dt - \omega_i dx^i\right)^2 + \frac{\gamma_{ij}}{\lambda} dx^i dx^j \tag{21}$$

We will also define for future use:

$$h_{ij} = \lambda^{-1}\gamma_{ij} \tag{22}$$

$$\lambda = e^{2U} \tag{23}$$

In particular for Schwarzschild spacetime in the Schwarzschild coordinate chart

$$\xi = \partial_t \tag{24}$$

it gives

$$\lambda = -\xi\xi = 1 - \frac{r_s}{r} \tag{25}$$

and the twist vanishes in the spacelike coordinates (i=1,2,3), which tells us that ξ is orthogonal to the spacelike hypersurfaces and that there is no rotation.

Now, having a timelike Killing field it means that we have a stationary spacetime.

For now, let us assume that the Killing field is irrotational

$$\omega_i = 0 \tag{26}$$

Then spacetime is static, and the Killing observers (four-velocities parallel to the timelike Killing field) are also static, since

$$dx^\mu = 0 \tag{27}$$

But we must normalize the four-velocity properly:

$$u^\mu = \lambda^{-1/2}\xi^\mu \tag{28}$$

Thus

$$\lambda^{-1/2}_{Schwarzschild} = \frac{1}{\sqrt{1 - \frac{r_s}{r}}} = \gamma_r \tag{29}$$

This is admittedly a lot of conceptual machinery just for the gravitational Lorentz gamma, but now we have a more general framework to work in, and find that in general this quantity is the inverse-norm of our timelike Killing field. Without the inversion, the quantity $\lambda^{1/2}$ is the gravitational redshift.

The four-acceleration of the Killing observers given by a covariant derivative in our static case simplifies to just:

$$a^{\mu} = U^{;\mu} = g^{\mu\nu}\left(\log\lambda^{1/2}\right)_{,\nu} \tag{30}$$

In the Schwarzschild case, everything but the r-component vanishes, and we can put it in corresponding orthonormal basis rather the coordinate vector:

$$\partial_r \left(\hat{e}_r = \sqrt{1-\frac{r_s}{r}}\partial_r\right) \tag{31}$$

$$a^r = \left(1-\frac{r_s}{r}\right)\frac{1}{2}\frac{r_s}{r^2}\frac{1}{1-\frac{r_s}{r}} = \frac{r_s}{2r^2} \tag{32}$$

Thus

$$a = \frac{r_s}{2r^2}\frac{1}{\sqrt{1-\frac{r_s}{r}}}\hat{e}_r = g_r\hat{e}_r \tag{33}$$

showing the correct gravitational acceleration: the proper acceleration of the Killing observers is g_r into the outwardly radial direction.

3.1.1. Conclusion

We have just seen that the following are all related:

- gravitational acceleration is related to gravitational gamma
- redshift is related to timelike Killing field

Similar but more complicated situation applies to stationary spacetime. While we can do a backwards derivation of Schwarzschild spacetime using an additional free-falling velocity

$$v_r = \sqrt{\frac{r_s}{r}} \tag{34}$$

the implicit assumption is that there is that space is spherically symmetric, and thus has only one degree of freedom (radial) left, which this additional field determines.

3.2. The gravitational action

Taking t=0 hypersurface, the symmetry of the spacetime under time inversion means that the extrinsic curvature is zero, in which case the Gauss-Codazzi equations simplify to:

$$R^{a}_{bcd} =^{h} R^{a}_{bcd} \tag{35}$$

$$R^{0}_{bcd} = 0 \tag{36}$$

$$R^{0}_{i0j} = -{}^{h}U_{;i;j} - \left({}^{h}U_{;i}\right)\left({}^{h}U_{;j}\right) \tag{37}$$

where the superscript h denotes that the quantity belongs to spatial hyperslice and should be calculated using the spatial metric h alone.

Contracting all the way down to the Ricci scalar:

$$R =^h R - 2\left[{}^hU_{;i}^{;i} + \left({}^hU^{;i}\right)\left({}^hU_{;i}\right)\right] \tag{38}$$

The first term in the bracket is a Laplacian

$$2U = \log\lambda \tag{39}$$

and the metric determinants are

$$g = -\lambda h = -\gamma \tag{40}$$

so in terms of the metric:

$$\gamma_{ij}\left(\frac{1}{\lambda}\right)h_{ij} \tag{41}$$

the representation of the Einstein-Hilbert action will be

$$S\ \alpha \int R\sqrt{-g} \tag{42}$$

$$S = \int\left({}^\gamma R -^\gamma \nabla^2\left(\log\lambda\right) - \frac{1}{2}\frac{\gamma_{ij}d\lambda^i d\lambda^j}{\lambda^2}\right)\sqrt{\gamma} \tag{43}$$

3.2.1. Conclusion

From above we see, that gravitational action of a static spacetime is equivalent to the action of a 3-dimensional spacelike manifold minimally coupled to a scalar field.

The same should be true for more general stationary spacetimes as well, just with an additional coupling to a vector field related to the twist above.

3.3. Rindler's transformation

From above description we may conclude, that stationary Killing observer behave as he would be accelerated against freefaling surroundings according to formula (33). Let us investigate these results and check if surroundings really may be treated as accelerated. To check it we will change reference frame and rewrite formulas using vector notation.

Let us consider flat Minkowski spacetime. Let us write Rindler's transformation for some body, accelerated at some stationary observer reference frame:

$$v\gamma = a\tau_{observ} \tag{44}$$

$$\frac{v}{a} = \tau \tag{45}$$

Let us imagine that in this flat spacetime, considered body moves with "v" and "a" values equal to free falling velocity and gravitational acceleration. We obtain:

$$\tau = \frac{v_r}{g_r} = 2r \cdot \sqrt{\frac{r}{r_s}} \cdot \sqrt{1 - \frac{r_s}{r}} \tag{46}$$

Now, we calculate derivative with respect to the radial variable and make simple transformations:

$$\frac{d\tau}{dr} = \frac{\sqrt{\frac{r_s}{r}}}{\sqrt{1 - \frac{r_s}{r}}} = v_r \cdot \gamma_r \tag{47}$$

$$d\tau^2 = dr^2\gamma_r^2 - dr^2 \tag{48}$$

Let us write Minkowski metric for stationary observer with radial coordinates:

$$d\tau_{observ}^2 = d\tau^2 + dr^2 + r^2 d\varphi^2 \tag{49}$$

Substituting (49) to (50) we obtain geodesics formula for stationary Killing observer (in some r distance to source of gravity):

$$0 = d\tau_{observ}^2 - dr^2\gamma_r^2 - r^2 d\varphi^2 \tag{50}$$

where for Killing observer's proper time and dt in infinity we have relationship:

$$d\tau_{observ} = dt \cdot \frac{1}{\gamma_r} \tag{51}$$

We recognize in above regular Schwarzschild metric for local Killing observer. Spacetime just get curved - accelerated body (considered locally) may be then recognized as photon.

3.3.1. Conclusion

Above transformation, formulas (29) and (33) drive to important conclusion: spacetime curvature may be explained by local attraction of flat spacetime according to scalar field equal to:

$$\frac{1}{\gamma_r} = \sqrt{1 - \frac{r_s}{r}} \tag{52}$$

We are used to explaining gravity as interaction between bodies what is utilized in quantum gravity theories. However, we have just derived different explanation.

Gravity tend to be explained as interaction with photons (local acceleration of surrounding spacetime) what makes the spacetime curved. Curved spacetime then affects paths of the bodies.

To show hypothetical implications of above let us follow [4] and assign to spacetime rest energy equal to Planck Energy. In considered case kinetic energy (and potential energy up to the sign) will be expressed with formula:

$$E_m = E_{planck} \cdot \left(\frac{1}{\sqrt{1 - \frac{r_s}{r}}} - 1 \right) \tag{53}$$

If we measure above for Planck length, for tiny masses $l_{planck} >> r_s$ we may use Maclaurin's expansion:

$$\lim_{r \to l_{Planck}} E_m \approx E_{planck} \cdot \frac{r_s}{2l_{planck}} = \frac{c^4 r_s}{2G} \tag{54}$$

obtaining valid rest energy formula (quantum for considered field).

4. Electromagnetic Tensor

In last section we have just discovered, that we may consider photon acceleration as reason for spacetime curvature increase. However, above needs urgent explanation for relation between spacetime and electromagnetism. Below I propose such explanation.

4.1. Disturbances of spacetime isometry

Killing vector X^b by definition [2] satisfies:

$$g^{bc} X_{c;ab} - R_{ab} X^b = 0 \tag{55}$$

$$X_{a;bc} = R_{abcd} X^d \tag{56}$$

$$X^{a;b}_{;b} + R^a_c X^c = 0 \tag{57}$$

Thus for Killing field above:

$$R^{\alpha}_{\beta\gamma\sigma} \xi^{\sigma} = \xi^{\sigma}_{;\beta;\gamma} \tag{58}$$

Therefore:

$$\xi^{;\beta}_{\alpha;\beta} = -R_{\alpha\beta} \xi^{\beta} \tag{59}$$

so the timelike Killing field is intimately connected to spacetime curvature. Defining a convenient F and using the defining property of Killing fields

$$\xi_{\alpha;\beta} + \xi_{\beta;\alpha} = 0 \tag{60}$$

we obtain

$$F_{\alpha\beta} = \xi_{[\beta;\alpha]} = \frac{1}{2} \left(\xi_{\beta;\alpha} - \xi_{\alpha;\beta} \right) = -\xi_{\alpha;\beta} \tag{61}$$

therefore

$$F^{;\beta}_{\alpha\beta} = R_{\alpha\beta} \xi^{\beta} \tag{62}$$

This looks like the standard electromagnetic field tensor

$$F^{;\beta}_{\alpha\beta} = A_{\beta;\alpha} - A_{\alpha;\beta} \tag{63}$$

which couples to four-current J through Maxwell's equation

$$F^{\alpha\beta}_{;\beta} = 4\pi J^{\alpha} \tag{64}$$

In a vacuum, the Ricci tensor vanishes, and the Killing field ξ seems to be like the electromagnetic four-potential A that acts for electromagnetism in source-free regions, in the Lorentz gauge

$$A^{\alpha}_{;\alpha} = 0 \tag{65}$$

which automatically satisfies all Maxwell's equations. It is notable that the comparison Maxwell's equations makes it natural to try to define a four-current

$$J^{\alpha} = R^{\alpha}_{\beta}\xi^{\beta} \tag{66}$$

and integrating this current (up to some factors) gives the *Komar mass* of the stationary spacetime.
Physical meaning of the tensor F should be explained as dislocating, local disturbance in spacetime isometry.

4.2. Maxwell equations for Minkowski spacetime

For better understanding of above results let us express it with vector notation. Let us define scalar potential equal to:

$$\frac{c}{\gamma_E} = c\sqrt{1 - \frac{l_{planck}}{r}} \tag{67}$$

We define following vector fields:

$$\vec{T} = \frac{c}{\gamma_E} \cdot \hat{e}_y \tag{68}$$

$$\vec{A} = -\nabla \frac{c}{\gamma_E} \times \hat{e}_y = -\nabla \times \vec{T} \tag{69}$$

$$\vec{U} = \nabla\left(r \cdot v_E\right) \times \hat{e}_x \tag{70}$$

$$\vec{\Omega} = \nabla \times \vec{U} = \frac{d\hat{e}_y}{dt} \tag{71}$$

Utilizing relations between above fields we obtain:

$$\frac{\gamma_E}{c} \cdot \frac{d\vec{T}}{dt} = \frac{\gamma_E}{c}\frac{c}{\gamma_E} \cdot \frac{d\hat{e}_y}{dt} = \vec{\Omega} \tag{72}$$

$$\nabla \times \vec{A} = \frac{\gamma_E}{c} \cdot \frac{d\vec{\Omega}}{dt} \tag{73}$$

where last relation we obtain by analogy from (33).

After simple transformations we derive d'Alambertian:

$$\frac{1}{c^2} \cdot \frac{\partial^2 \vec{A}}{\partial \tau^2} - \nabla^2 \vec{A} = 0 \tag{74}$$

In last section we have just derived the same wave with Tensor notation. In this perspective time flow axis is still orthogonal to spatial axis but acts as rotary field. Therfore we may still consider both as Minkowski spacetime with orthogonal space and time.

If we accept that light might be explained as local spacetime anizometry propagating through spacetime [4] then proposed tensor based on Killing fields opens new areas for research for Quantum Field theories.

4.2.1. Conclusion - Electrodynamics

To show hypothetical implications of above let us follow [4] and consider rest energy equal to Planck Energy. Let us use just defined scalar field but express it with time quantities:

$$\frac{1}{\gamma_E} = \sqrt{1 - \frac{t_{planck}}{t}} \tag{75}$$

For t >> t_{planck} we might use Maclaurin's expansion introducing some new quantity of energy E:

$$E = E_{planck} \left(\frac{1}{\sqrt{1 - \frac{t_{planck}}{t}}} - 1 \right) \approx E_{planck} \frac{t_{planck}}{2t} \tag{76}$$

$$2E \approx \hbar\omega \tag{77}$$

where:

$$\omega = \frac{1}{t} \tag{78}$$

We see, that new scalar field might represent the presence of Electromagnetic field, what will be confirmed farther. We should notice, that hypothetical photon energy formula introduced above may be tested for pulsations close to Planck pulsation $\omega_{planck} = 1/t_{planck}$ treating ω_{planck} as the highest limit of pulsation.

While considering a photon described by the commonly used energy formula "hv" we obtain problem for measurement at Planck pulsation. Such photon would need both positive and negative polarities of both the electric and magnetic fields ideally showing both and neither at the same time.

Just introduced formula based on (76) does not crash at Planck time scales.

Let us merge two energies defined in (54) (e.g. in annihilation event creating one boson) and express two new particles with just introduced new field:

$$\hbar\omega = 2E_m \tag{79}$$

when:

$$\frac{r_s}{l_{planck}} = \frac{t_{planck}}{t} \tag{80}$$

4.2.2. Conclusion - Electrostatics

We should notice that relation between just introduced pulsation (78) and period T is:

$$\omega = \frac{1}{t} = \frac{2\pi}{T} \tag{81}$$

Let us show, that using above relation we may derive electrostatic potential expressed by fine structure constant α . Let us define interaction based on following scalar field:

$$\frac{1}{\gamma_Q} = \sqrt{1 - \frac{l_{planck}}{2\pi\, r}} \tag{82}$$

By analogy to mass definition (53) we define quantity:

$$E_{planck}\left(\frac{1}{\sqrt{1 - \frac{t_{planck}}{2\pi\, t_{planck}}}} - 1\right) \approx E_{planck} \cdot 4\pi\alpha \tag{83}$$

where alpha is fine structure constant.

Considering above hypothetical energy quantity in defined scalar field we obtain electrostatic potential for elementary charges (expressed with natural units):

$$4\pi\alpha \cdot E_{planck}\left(\frac{1}{\sqrt{1 - \frac{l_{planck}}{2\pi\, r}}} - 1\right) \approx 4\pi\alpha \cdot E_{planck} \cdot \frac{l_{planck}}{4\pi\, r} = \frac{\hbar c}{r} \cdot \alpha \tag{84}$$

Considering above derivation we may assign elementary charge phenomena with some rotation on radius close to Planck Length. Above might be treated as onfirmation of existence an additional, small scale dimension requested by Kaluza-Kalin theory or String theory.

5. Elementary particles and Kerr metric

Let us recall relation between ergosphere radius "r_s" and event horizon radius "rh" in Kerr metric [5] for spinning black hole.

$$2r_h = r_s + \sqrt{r_s^2 - 4\alpha^2} \tag{85}$$

where alpha is given by relation between black hole mass "M" and angular momentum "J".

$$J = \alpha \cdot cM \tag{86}$$

Now, let us consider imaginary angular momentum value as the result of rotation on imaginary radius "t"

$$\alpha = it \tag{87}$$

$$J = ict \cdot M \tag{88}$$

By simple transformation we obtain postulated in previous sections relation between ergosphere and horizon radius.

$$\frac{t}{r_h} = \sqrt{1 - \frac{r_s}{r_h}} \tag{89}$$

Substituting "r_h" with Planck length and keeping in mind that "t" was the distance within imaginary axis (time axis) we obtain the same relation that was utilized in formula (54) for elementary particle rest energy:

$$\frac{1}{\gamma} = \sqrt{1 - \frac{r_s}{l_{planck}}} \tag{90}$$

We should notice, that for imaginary value of angular momentum and ergosphere radius smaller that event horizon radius, just described spinning black hole does not evaporate immediately as usual.

Above drives to description of small, spinning Kerr black holes living long enough to behave as elementary particles.

In Quantum Mechanic we consider elementary particle by wave function, that express rotation at complex unit circle:

$$e^{ix} = \cos(x) - i\sin(x) \tag{91}$$

Now, we may try to link above rotation with spinning Kerr black hole where event horizon radius is equal to Planck length and rotation is around imaginary radius.

According to Quantum Mechanics postulate, considered imaginary radius - by its influence on "r_s" value - appears to be responsible for energy of the body.

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